

EXAMPLES OF HYPERBOLIC HYPERSURFACES OF LOW DEGREE IN PROJECTIVE SPACES

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Abstract

We construct families of hyperbolic hypersurfaces of degree $2n$ in the projective space $\mathbb{P}^n(\mathbb{C})$ for $3 \leq n \leq 6$.

Keywords: Kobayashi conjecture, hyperbolicity, Brody Lemma, Nevanlinna Theory

1 Introduction and the main result

The Kobayashi conjecture states that a generic hypersurface $X_d \subset \mathbb{P}^n(\mathbb{C})$ of degree $d \geq 2n - 1$ is hyperbolic. It is proved by Demailly and El Goul [5] for $n = 3$ and a very generic surface of degree at least 21. In [17], Păun improved the degree to 18. In $\mathbb{P}^4(\mathbb{C})$, Rousseau [18] was able to show that a generic three-fold of degree at least 593 contains no Zariski-dense entire curve, a result from which hyperbolicity follows, after removing divisorial components [7]. In $\mathbb{P}^n(\mathbb{C})$, for any n and for $d \geq 2^{(n-1)^5}$, Diverio, Merker and Rousseau [6] established algebraic degeneracy of entire curves in X_d . Also, Siu [21] proposed a positive answer for arbitrary n and degree $d = d(n) \gg 1$ very large. Most recently, Demailly [4] has announced a strategy that is expected to attain Kobayashi's conjecture for *very* generic hypersurfaces of degree $d \geq 2n$.

Concurrently, many authors tried to find examples of hyperbolic hypersurfaces of degree as low as possible. The first example of a *compact* Kobayashi hyperbolic manifold of dimension 2 is a hypersurface in $\mathbb{P}^3(\mathbb{C})$ constructed by Brody and Green [2]. Also, the first examples in all higher dimensions $n - 1 \geq 3$ were discovered by Masuda and Noguchi [16], with degree large. So far, the best degree asymptotic is the square of dimension, given by Siu and Yeung [22] with $d = 16(n - 1)^2$ and by Shiffman and Zaidenberg [19] with $d = 4(n - 1)^2$. In $\mathbb{P}^3(\mathbb{C})$ many examples of low degree were given (see the reference of [23]). The lowest degree found up to date is 6, given by Duval [9]. There are not so many examples of low degree hyperbolic hypersurfaces in $\mathbb{P}^4(\mathbb{C})$. We mention here an example of a hypersurface of degree 16 constructed by Fujimoto [13]. Various examples in $\mathbb{P}^5(\mathbb{C})$ and $\mathbb{P}^6(\mathbb{C})$ only appear in the cases of arbitrary dimension mentioned above.

Before going to introduce the main result, we need some notations and conventions. A family of hyperplanes $\{H_i\}_{1 \leq i \leq q}$ with $q \geq n + 1$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in *general position* if any $n + 1$ hyperplanes in this family have empty intersection. A hypersurface S in $\mathbb{P}^n(\mathbb{C})$ is said to be in *general position with respect to* $\{H_i\}_{1 \leq i \leq q}$ if it avoids all intersection points of n hyperplanes in $\{H_i\}_{1 \leq i \leq q}$, namely if:

$$S \cap \left(\bigcap_{i \in I} H_i \right) = \emptyset, \quad \forall I \subset \{1, \dots, q\}, |I| = n.$$

Now assume that $\{H_i\}_{1 \leq i \leq q}$ is a family of hyperplanes of $\mathbb{P}^n(\mathbb{C})$ ($n \geq 2$) in general position. Let $\{H_i\}_{i \in I}$ be a subfamily of $n + 2$ hyperplanes. Take a partition $I = J \cup K$ such that $|J|, |K| \geq 2$. Then there exists a unique hyperplane H_{JK} containing $\bigcap_{j \in J} H_j$ and $\bigcap_{k \in K} H_k$. We call H_{JK} a *diagonal hyperplane* of $\{H_i\}_{i \in I}$. The family $\{H_i\}_{1 \leq i \leq q}$ is said to be *generic* if, for all disjoint subsets I, J, J_1, \dots, J_k of $\{1, \dots, q\}$ such that $|I|, |J_i| \geq 2$ and $|I| + |J_i| = n + 2$, $1 \leq i \leq k$, for every subset $\{i_1, \dots, i_l\}$ of I , the intersection between the $|J|$ hyperplanes $H_j, j \in J$, the k diagonal hyperplanes $H_{IJ_1}, \dots, H_{IJ_k}$, and the l hyperplanes H_{i_1}, \dots, H_{i_l} is a linear subspace of codimension $\min\{k + l, |I|\} + |J|$, with the convention that when $\min\{k + l, |I|\} + |J| > n$, this intersection is empty. Such a generic condition naturally appears in our constructions, and it has the virtue of being preserved when passing to

smaller-dimensional subspaces

Our aim in this article is to prove that, for $3 \leq n \leq 6$, a small deformation of a union of generic $2n$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ is hyperbolic.

Main Theorem. *Let n be an integer number in $\{3, 4, 5, 6\}$. Let $\{H_i\}_{1 \leq i \leq 2n}$ be a family of $2n$ generic hyperplanes in $\mathbb{P}^n(\mathbb{C})$, where $H_i = \{h_i = 0\}$. Then there exists a hypersurface $S = \{s = 0\}$ of degree $2n$ in general position with respect to $\{H_i\}_{1 \leq i \leq 2n}$ such that the hypersurface*

$$\Sigma_\epsilon = \{\epsilon s + \Pi_{i=1}^{2n} h_i = 0\}$$

is hyperbolic for sufficiently small complex $\epsilon \neq 0$.

Our proof is based on the technique of Duval [10] in the case $n = 3$. By the deformation method of Zaidenberg and Shiffman[20], the problem reduces to finding a hypersurface S such that all complements of the form

$$\cap_{i \in I} H_i \setminus (\cup_{j \notin I} H_j \setminus S)$$

are hyperbolic. This situation is very close to Theorem 2.5. To create such S , we proceed by deformation in order to allow points of intersection of S with more and more linear subspaces coming from the family $\{H_i\}_{1 \leq i \leq 2n}$.

Acknowledgments

This is a part of my Ph.D. thesis at Université Paris-Sud. I would like to gratefully thank my thesis advisor Julien Duval for his support, his very careful readings and many inspiring discussions on the subject. I am specially thankful to my thesis co-advisor Joël Merker for his encouragements, his prompt helps in \LaTeX and his comments that greatly improved the manuscript. I would like to thank Junjiro Noguchi for his bibliographical help. Finally, I acknowledge support from Hue University - College of Education.

2 Notations and preparation

2.1 Brody Lemma

Let X be a compact complex manifold equipped with a hermitian metric $\|\cdot\|$. By an *entire curve* in X we mean a nonconstant holomorphic map $f : \mathbb{C} \rightarrow X$. A *Brody curve* in X is an entire curve $f : \mathbb{C} \rightarrow X$ such that $\|f'\|$ is bounded. Brody curves arise as limits of sequences of holomorphic maps as follows (see [1]).

Brody Lemma. *Let $f_n : \mathbb{D} \rightarrow X$ be a sequence of holomorphic maps from the unit disk to a compact complex manifold X . If $\|f'_n(0)\| \rightarrow \infty$ as $n \rightarrow \infty$, then there exist a point $a \in \mathbb{D}$, a sequence (a_n) converging to a , and a decreasing sequence (r_n) of positive real numbers converging to 0 such that the sequence of maps*

$$z \rightarrow f_n(a_n + r_n z)$$

converges toward a Brody curve, after extracting a subsequence.

From an entire curve in X , the Brody Lemma also produces a Brody curve in X . A second consequence is a well-known characterization of Kobayashi hyperbolicity.

Brody Criterion. *A compact complex manifold X is Kobayashi hyperbolic if and only if it contains no entire curve (or no Brody curve).*

We shall repeatedly use the Brody Lemma under the following form.

Sequences of entire curves. *Let X be a compact complex manifold and let (f_n) be a sequence of entire curves in X . Then there exist a sequence of reparameterizations $r_n : \mathbb{C} \rightarrow \mathbb{C}$ and a subsequence of $(f_n \circ r_n)$ which converges towards an entire curve (or Brody curve).*

2.2 Nevanlinna theory and some applications

We recall some facts from Nevanlinna theory in the projective space $\mathbb{P}^n(\mathbb{C})$. Let $E = \sum \mu_\nu a_\nu$ be a divisor on \mathbb{C} and let $k \in \mathbb{N} \cup \{\infty\}$. Summing the k -truncated degrees of the divisor on disks by

$$n^{[k]}(t, E) := \sum_{|a_\nu| < t} \min\{k, \mu_\nu\} \quad (t > 0),$$

the *truncated counting function at level k* of E is defined by

$$N^{[k]}(r, E) := \int_1^r \frac{n^{[k]}(t, E)}{t} dt \quad (r > 1).$$

When $k = \infty$, we write $n(t, E)$, $N(r, E)$ instead of $n^{[\infty]}(t, E)$, $N^{[\infty]}(r, E)$. We denote the zero divisor of a nonzero meromorphic function φ by $(\varphi)_0$. Let $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ be an entire curve having a reduced representation $f = [f_0 : \cdots : f_n]$ in the homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{P}^n(\mathbb{C})$. Let $D = \{Q = 0\}$ be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ defined by a homogeneous polynomial $Q \in \mathbb{C}[z_0, \dots, z_n]$ of degree $d \geq 1$. If $f(\mathbb{C}) \not\subset D$, we define the *truncated counting function* of f with respect to D as

$$N_f^{[k]}(r, D) := N^{[k]}(r, (Q \circ f)_0).$$

The *proximity function* of f for the divisor D is defined as

$$m_f(r, D) := \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d \|Q\|}{|Q(f)(re^{i\theta})|} \frac{d\theta}{2\pi},$$

where $\|Q\|$ is the maximum absolute value of the coefficients of Q and

$$\|f(z)\| = \max\{|f_0(z)|, \dots, |f_n(z)|\}.$$

Since $|Q(f)| \leq \|Q\| \cdot \|f\|^d$, one has $m_f(r, D) \geq 0$. Finally, the *Cartan order function* of f is defined by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta.$$

It is known that [11] if f is a Brody curve, then its order

$$\rho_f := \limsup_{r \rightarrow +\infty} \frac{T_f(r)}{\log r}$$

is bounded from above by 2. Furthermore, Eremenko [11] showed the following.

Theorem 2.1. *If $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ is a Brody curve omitting n hyperplanes in general position, then it is of order 1.*

Consequently, we have the following theorem.

Theorem 2.2. *If $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ is a Brody curve avoiding the first n coordinate hyperplanes $\{z_i = 0\}_{i=0}^{n-1}$, then it has a reduced representation of the form*

$$[1 : e^{\lambda_1 z + \mu_1} : \cdots : e^{\lambda_{n-1} z + \mu_{n-1}} : g],$$

where g is an entire function and λ_i, μ_i are constants. If f also avoids the remaining coordinate hyperplane $\{z_n = 0\}$, then g is of the form $e^{\lambda_n z + \mu_n}$.

The core of Nevanlinna theory consists of two main theorems.

First Main Theorem. *Let $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let D be a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$ such that $f(\mathbb{C}) \not\subset D$. Then for every $r > 1$, the following holds*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1),$$

hence

$$N_f(r, D) \leq dT_f(r) + O(1). \tag{2.1}$$

A holomorphic curve $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ is said to be *linearly nondegenerate* if its image is not contained in any hyperplane. For non-negatively valued functions $\varphi(r)$, $\psi(r)$, we write

$$\varphi(r) \leq \psi(r) \parallel$$

if this inequality holds outside a Borel subset E of $(0, +\infty)$ of finite Lebesgue measure. Next is the Second Main Theorem of Cartan [3].

Second Main Theorem. *Let $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ and let $\{H_i\}_{1 \leq i \leq q}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then the following estimate holds:*

$$(q - n - 1) T_f(r) \leq \sum_{i=1}^q N_f^{[n]}(r, H_i) + S_f(r),$$

where $S_f(r)$ is a small term compared with $T_f(r)$

$$S_f(r) = o(T_f(r)) \parallel.$$

The next three theorems can be deduced from the Second Main Theorem.

Theorem 2.3. *Let $\{H_i\}_{1 \leq i \leq n+2}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ with $n \geq 2$. If $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C}) \setminus \cup_{i=1}^{n+2} H_i$ is an entire curve, then its image lies in one of the diagonal hyperplanes of $\{H_i\}_{1 \leq i \leq n+2}$.*

The following strengthened theorem is due to Dufresnoy [8].

Theorem 2.4. *If a holomorphic map $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ has its image in the complement of $n + p$ hyperplanes H_1, \dots, H_{n+p} in general position, then this image is contained in a linear subspace of dimension $\left\lfloor \frac{n}{p} \right\rfloor$.*

As a consequence, we have the classical generalization of Picard's Theorem (case $n = 1$), due to Fujimoto [12] (see also [14]).

Theorem 2.5. *The complement of a collection of $2n + 1$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ is hyperbolic.*

For hyperplanes that are not in general position, we have the following result (see [15], Theorem 3.10.15).

Theorem 2.6. *Let $\{H_i\}_{1 \leq i \leq q}$ be a family of $q \geq 3$ hyperplanes that are not in general position in $\mathbb{P}^n(\mathbb{C})$. If $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C}) \setminus \cup_{i=1}^q H_i$ is an entire curve, then its image lies in some hyperplane.*

3 Starting lemmas

Let us introduce some notations before going to other applications. Let $\{H_i\}_{1 \leq i \leq q}$ be a family of generic hyperplanes of $\mathbb{P}^n(\mathbb{C})$, where $H_i = \{h_i = 0\}$. For some integer $0 \leq k \leq n - 1$ and some subset $I_k = \{i_1, \dots, i_{n-k}\}$ of the index set $\{1, \dots, q\}$ having cardinality $n - k$, the linear subspace $P_{k, I_k} = \cap_{i \in I_k} H_i \simeq \mathbb{P}^k(\mathbb{C})$ is called a *subspace of dimension k* . For a holomorphic mapping $f: \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$, we define

$$n_f(t, P_{k, I_k}) := \sum_{|z| < t, f(z) \in P_{k, I_k}} \min_{i \in I_k} \text{ord}_z(h_i \circ f) \quad (t > 0),$$

where we take the sum only for z in the preimage of P_{k, I_k} , and

$$N_f(r, P_{k, I_k}) := \int_1^r \frac{n_f(t, P_{k, I_k})}{t} dt \quad (r > 1). \quad (3.1)$$

We denote by P_{k,I_k}^* the complement $P_{k,I_k} \setminus (\cup_{i \notin I_k} H_i)$ which will be called a *star-subspace of dimension k* . We can also define $n_f(t, P_{k,I_k}^*)$ and $N_f(r, P_{k,I_k}^*)$. Assume now $q = 2n + 1 + m$ with $m \geq 0$. Consider complements of the form

$$\mathbb{P}^n(\mathbb{C}) \setminus (\cup_{i=1}^{2n+1+m} H_i \setminus A_{m,n}), \quad (3.2)$$

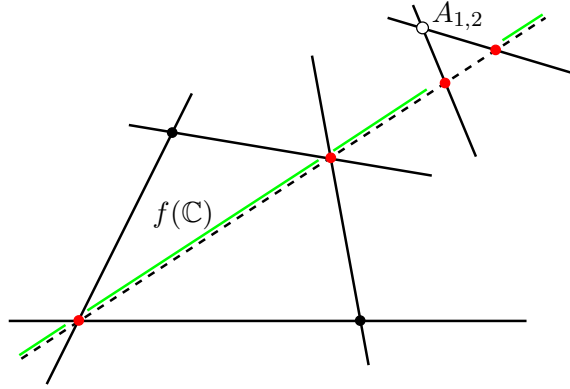
where $A_{m,n}$ is a set of at most m elements of the form P_{k,I_k}^* ($0 \leq k \leq n-2$). We note that if $m = 0$, these complements are hyperbolic by Theorem 2.5.

In $\mathbb{P}^2(\mathbb{C})$, a union of lines $\cup_{i=1}^q H_i$ is in general position if any three lines have empty intersection, and it is generic if in addition any three intersection points between three distinct pairs of lines are not collinear.

Lemma 3.1. *In $\mathbb{P}^2(\mathbb{C})$, if $m \leq 3$, all complements of the form (3.2) are hyperbolic.*

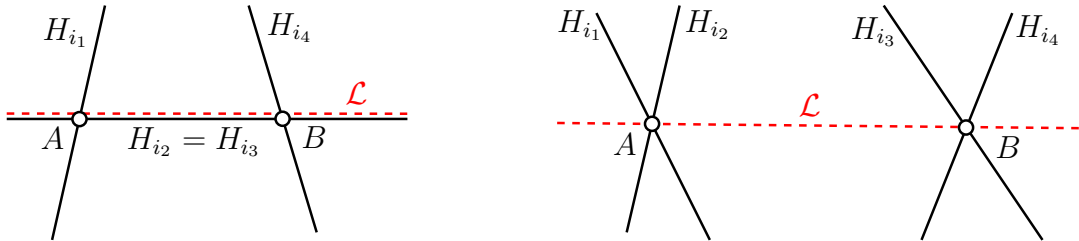
Proof. Without loss of generality, we can assume that $A_{m,2}$ is a set of m distinct points belonging to $\cup_{1 \leq i_1 < i_2 \leq 5+m} H_{i_1} \cap H_{i_2}$.

When $m = 1$, an entire curve $f: \mathbb{C} \rightarrow \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^6 H_i \setminus A_{1,2})$, if it exists, must avoid at least four lines.



By Theorem 2.3, its image lies in a diagonal line, which does not contain the intersection point of the two remaining lines by the generic condition. Hence, f must be contained in the complement of four points in a line. By Picard's theorem, f is constant, which is contradiction.

When $m = 2$, $A_{2,2}$ is a set consisting of two points A, B , where $A = H_{i_1} \cap H_{i_2}$, $B = H_{i_3} \cap H_{i_4}$. We denote by I the index set $\{i_1, i_2, i_3, i_4\}$, which has three elements if both A and B belong to a single line H_i and which has four elements otherwise.



Let $f: \mathbb{C} \rightarrow \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^7 H_i \setminus A_{2,2})$ be an entire curve. If $z \in f^{-1}(A)$, we have

$$\begin{aligned} \text{ord}_z(h_{i_1} \circ f) &\geq 1, \\ \text{ord}_z(h_{i_2} \circ f) &\geq 1. \end{aligned}$$

This implies

$$\min \{\text{ord}_z(h_{i_1} \circ f), 2\} + \min \{\text{ord}_z(h_{i_2} \circ f), 2\} \leq 3 \min_{1 \leq j \leq 2} \text{ord}_z(h_{i_j} \circ f). \quad (3.3)$$

Hence, by summing this inequality

$$\sum_{|z| < t, f(z)=A} \min \{\text{ord}_z(h_{i_1} \circ f), 2\} + \sum_{|z| < t, f(z)=A} \min \{\text{ord}_z(h_{i_2} \circ f), 2\} \leq 3 \sum_{|z| < t, f(z)=A} \min_{1 \leq j \leq 2} \text{ord}_z(h_{i_j} \circ f). \quad (3.4)$$

Similarly for h_{i_3} , h_{i_4} and $z \in f^{-1}(B)$, we have

$$\sum_{|z| < t, f(z)=B} \min \{ \text{ord}_z(h_{i_3} \circ f), 2 \} + \sum_{|z| < t, f(z)=B} \min \{ \text{ord}_z(h_{i_4} \circ f), 2 \} \leq 3 \sum_{|z| < t, f(z)=B} \min_{3 \leq j \leq 4} \text{ord}_z(h_{i_j} \circ f). \quad (3.5)$$

By taking the sum of both sides of these inequalities and by integrating, we obtain

$$\sum_{i \in I} N_f^{[2]}(r, H_i) \leq 3 (N_f(r, A) + N_f(r, B)). \quad (3.6)$$

Now, let $\mathcal{L} = \{\ell = 0\}$ be the line passing through A and B . Since $\ell = \alpha_1 h_{i_1} + \alpha_2 h_{i_2} = \alpha_3 h_{i_3} + \alpha_4 h_{i_4}$ for some $\alpha_1, \dots, \alpha_4 \in \mathbb{C}$, the following inequalities hold

$$\begin{aligned} \min_{1 \leq j \leq 2} \text{ord}_z(h_{i_j} \circ f) &\leq \text{ord}_z(\ell \circ f) & (z \in f^{-1}(A)), \\ \min_{3 \leq j \leq 4} \text{ord}_z(h_{i_j} \circ f) &\leq \text{ord}_z(\ell \circ f) & (z \in f^{-1}(B)). \end{aligned} \quad (3.7)$$

Since $f^{-1}(A)$ and $f^{-1}(B)$ are two disjoint subsets of $f^{-1}(\mathcal{L})$, by taking the sum of both sides of these inequalities on discs and by integrating, we obtain

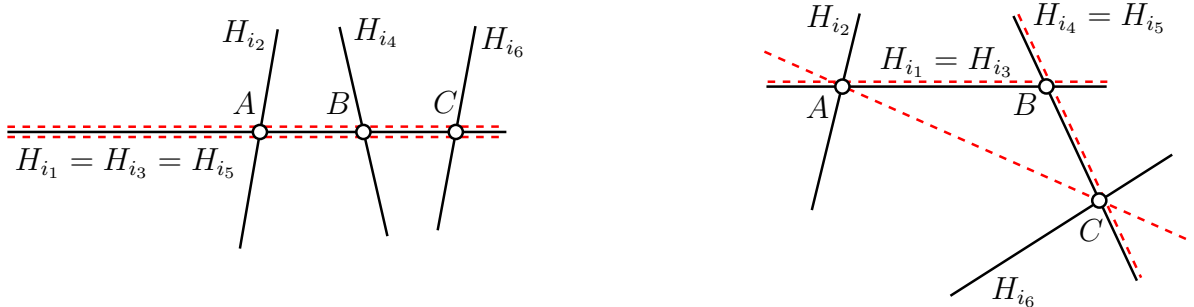
$$N_f(r, A) + N_f(r, B) \leq N_f(r, \mathcal{L}). \quad (3.8)$$

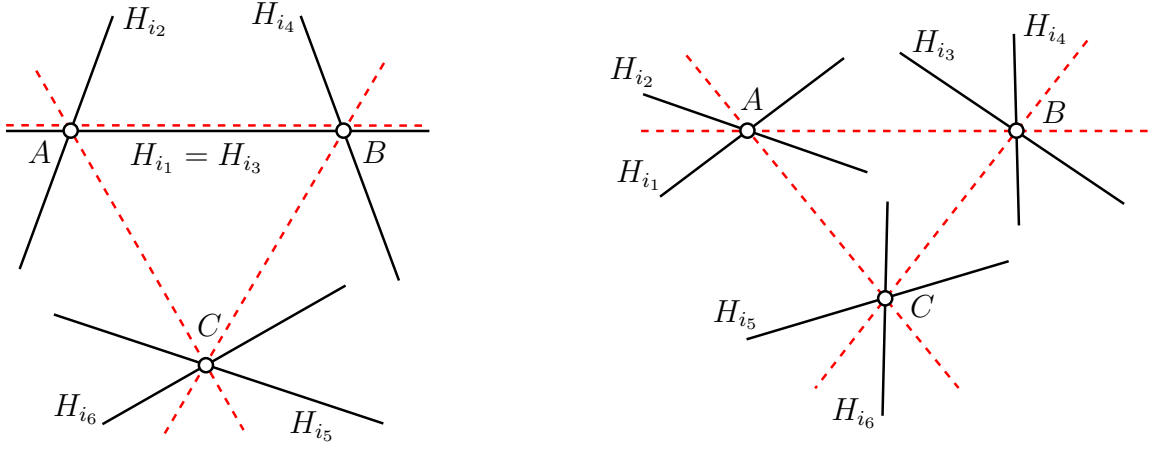
If f would be linearly nondegenerate, then starting from Cartan Second Main Theorem, and using (3.6), (3.8), we would get

$$\begin{aligned} 4T_f(r) &\leq \sum_{i=1}^7 N_f^{[2]}(r, H_i) + S_f(r) \\ &= \sum_{i \in I} N_f^{[2]}(r, H_i) + S_f(r) \\ &\leq 3 (N_f(r, A) + N_f(r, B)) + S_f(r) \\ &\leq 3 N_f(r, \mathcal{L}) + S_f(r) \\ &\stackrel{[\text{Use (2.1)}]}{\leq} 3T_f(r) + S_f(r), \end{aligned} \quad (3.9)$$

which is absurd. Thus, any entire curve $f : \mathbb{C} \rightarrow \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^7 H_i \setminus A_{2,2})$ must be contained in some line L . Furthermore, the number of points of intersection between L and $\cup_{i=1}^7 H_i \setminus \{A, B\}$ is at least 3 by the generic condition. By Picard's Theorem, this contradicts the assumption that f is nonconstant.

When $m = 3$, $A_{3,2}$ is a set consisting of three points A, B, C , where $A = H_{i_1} \cap H_{i_2}$, $B = H_{i_3} \cap H_{i_4}$, $C = H_{i_5} \cap H_{i_6}$. In this case, the index set $J = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ may contain 4, 5 or 6 elements.





Suppose to the contrary that there is an entire curve $f: \mathbb{C} \rightarrow \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^8 H_i \setminus A_{3,2})$. Similarly as above, cf. (3.4), (3.5), (3.6), we can show in all the four illustrated cases

$$\sum_{i \in J} N_f^{[2]}(r, H_i) \leq 3(N_f(r, A) + N_f(r, B) + N_f(r, C)).$$

Next, let $\mathcal{C} = \{c = 0\}$ be the degenerate cubic consisting of the three lines $AB = \{\ell_{AB} = 0\}$, $BC = \{\ell_{BC} = 0\}$, and $CA = \{\ell_{CA} = 0\}$. Similarly as in (3.7), we have

$$\begin{aligned} 2 \min_{1 \leq j \leq 2} \text{ord}_z(h_{i_j} \circ f) &\leq \text{ord}_z(\ell_{AB} \circ f) + \text{ord}_z(\ell_{CA} \circ f) \\ &= \text{ord}_z(c \circ f) \end{aligned} \quad (z \in f^{-1}(A)).$$

We also have two other inequalities for $h_{i_3}, h_{i_4}, z \in f^{-1}(B)$ and for $h_{i_5}, h_{i_6}, z \in f^{-1}(C)$. Summing these inequalities and integrating, we get

$$2(N_f(r, A) + N_f(r, B) + N_f(r, C)) \leq N_f(r, \mathcal{C}).$$

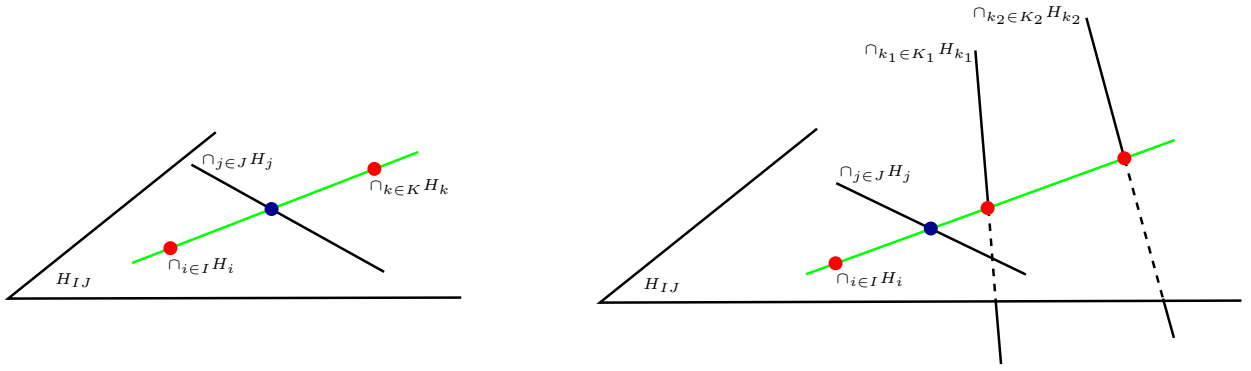
If the curve f is linearly nondegenerate, then by proceeding as we did in (3.9), we also get a contradiction.

$$\begin{aligned} 5T_f(r) &\leq \sum_{i=1}^8 N_f^{[2]}(r, H_i) + S_f(r) \\ &\leq 3(N_f(r, A) + N_f(r, B) + N_f(r, C)) + S_f(r) \\ &\leq \frac{3}{2}N_f(r, \mathcal{C}) + S_f(r) \\ &\leq \frac{9}{2}T_f(r) + S_f(r). \end{aligned}$$

Thus the curve f must be contained in some line. By analyzing the position of this line with respect to $\{H_i\}_{1 \leq i \leq 8} \setminus \{A, B, C\}$ and by using Picard's theorem, we conclude as above. \square

In $\mathbb{P}^3(\mathbb{C})$, the generic condition for the family of planes $\{H_i\}_{1 \leq i \leq q}$ excludes the following cases.

- (1) There are three disjoint subsets I, J, K of $\{1, \dots, q\}$ with $|I| = 3, |J| = 2, |K| = 3$ such that the diagonal (hyper)plane H_{IJ} contains the point $\cap_{k \in K} H_k$.
- (2) There are four disjoint subsets I, J, K_1, K_2 of $\{1, \dots, q\}$ with $|I| = 3, |J| = 2, |K_1| = |K_2| = 2$ such that the three points $(\cap_{k_1 \in K_1} H_{k_1}) \cap H_{IJ}$, $(\cap_{k_2 \in K_2} H_{k_2}) \cap H_{IJ}$ and $\cap_{i \in I} H_i$ are collinear.



Lemma 3.2. *In $\mathbb{P}^3(\mathbb{C})$, if $m \leq 2$, all complements of the form (3.2) are hyperbolic.*

Proof. Without loss of generality, we can assume that $A_{m,3}$ is a set of m elements belonging to:

$$\left(\bigcup_{1 \leq i_1 < i_2 \leq 7+m} (H_{i_1} \cap H_{i_2})^* \right) \bigcup \left(\bigcup_{1 \leq i_1 < i_2 < i_3 \leq 7+m} H_{i_1} \cap H_{i_2} \cap H_{i_3} \right).$$

Suppose to the contrary that there exists a Brody curve $f: \mathbb{C} \rightarrow \mathbb{P}^3(\mathbb{C}) \setminus (\bigcup_{i=1}^{7+m} H_i \setminus A_{m,3})$.

When $m = 1$, the curve f must avoid at least five planes. By Theorem 2.4, its image is contained in some line L . By the generic condition, the number of intersection points between L and $\bigcup_{i=1}^8 H_i \setminus A_{1,3}$ is at least 3. By Picard's theorem, f must be constant, which is a contradiction.

Next, we consider the case $m = 2$. If $A_{2,3} = \{l_1^*, l_2^*\}$ where l_1, l_2 are lines, then the curve f avoids five planes, say $\{H_i\}_{1 \leq i \leq 5}$. By Theorems 2.4 and 2.3, its image lands in some line \mathcal{L} , which is contained in a diagonal plane \mathcal{P} of the family $\{H_i\}_{1 \leq i \leq 5}$. We may assume that the plane \mathcal{P} passes through the point $H_1 \cap H_2 \cap H_3$ and contains the line $H_4 \cap H_5$. If the line \mathcal{L} does not pass through the point $H_1 \cap H_2 \cap H_3$, then it intersects $\{H_i\}_{1 \leq i \leq 3}$ in three distinct points, hence f is constant by Picard's theorem. Thus \mathcal{L} must pass through the point $H_1 \cap H_2 \cap H_3$. In the plane \mathcal{P} , the curve f can pass through the points $l_1 \cap \mathcal{P}$ and $l_2 \cap \mathcal{P}$. But by the generic condition, cf. (2) above, the three points $H_1 \cap H_2 \cap H_3, l_1 \cap \mathcal{P}, l_2 \cap \mathcal{P}$ are not collinear. Hence, $f(\mathbb{C})$ is contained in a complement of at least three points in the line \mathcal{L} , which is impossible by Picard's theorem.

Two substantial cases remain:

- (a) $A_{2,3} = \{A, l^*\}$, where A is a point and l is a line;
- (b) the set $A_{2,3}$ consists of two points.

We treat case (a). If both A and l are contained in some common plane H_i , then f avoids five planes. By Theorem 2.3, its image must be contained in some diagonal plane, which does not contain the point A by the generic condition. Hence f must avoid seven planes in general position, which is absurd by Theorem 2.5. Thus, we can assume that $A = H_1 \cap H_2 \cap H_3$ and $l^* = (H_4 \cap H_5) \setminus \bigcup_{i \neq 4,5} H_i$. Hence f avoids four planes H_i ($6 \leq i \leq 9$).

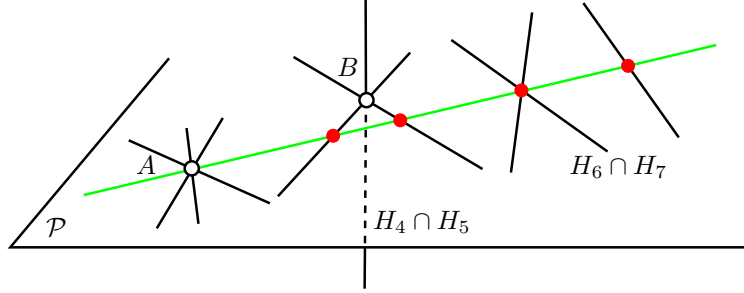
First, we show that f is linearly nondegenerate. Suppose to the contrary that $f(\mathbb{C})$ is contained in some plane \mathcal{P} . If $A \notin \mathcal{P}$, then f also avoids H_1, H_2, H_3 , which is impossible by Theorem 2.5. Hence the plane \mathcal{P} must pass through the point A . If $f(\mathbb{C})$ is contained in some line $\mathcal{L} \subset \mathcal{P}$, then \mathcal{L} must also pass through A , for the same reason. Note that the number of intersection points between \mathcal{L} and $\{H_i\}_{6 \leq i \leq 9}$ is at least 2, and it equals 2 only if either \mathcal{L} passes through some point $H_{i_1} \cap H_{i_2} \cap H_{i_3}$ ($6 \leq i_1 < i_2 < i_3 \leq 9$) or \mathcal{L} intersects two lines $H_{i_1} \cap H_{i_2}, H_{i_3} \cap H_{i_4}$ ($\{i_1, i_2, i_3, i_4\} = \{6, 7, 8, 9\}$). If \mathcal{L} has empty intersection with $H_4 \cap H_5$, then f avoids at least four points in the line \mathcal{L} , hence it is constant. If \mathcal{L} intersects $H_4 \cap H_5$, then by considering the diagonal plane passing through A and containing $H_4 \cap H_5$, the two cases where $|\mathcal{L} \cap \{H_i\}_{6 \leq i \leq 9}| = 2$ are excluded by the generic condition. Thus f always avoids three distinct points in \mathcal{L} , hence it is constant.

Consequently, we can assume that f does not land in any line in the plane \mathcal{P} . There are two possible positions of \mathcal{P} :

- (a1) it is a diagonal plane containing A and some line in $\bigcup_{6 \leq i_1 < i_2 \leq 9} H_{i_1} \cap H_{i_2}$;

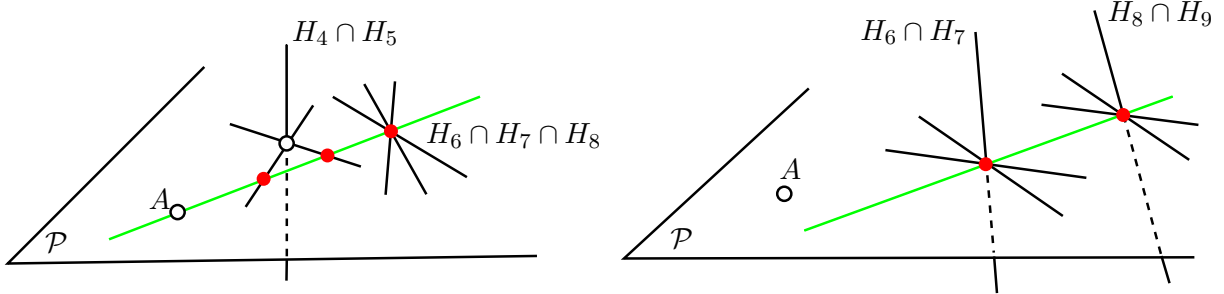
(a2) it does not contain any line in $\cup_{6 \leq i_1 < i_2 \leq 9} H_{i_1} \cap H_{i_2}$.

In case (a1), assume that \mathcal{P} contains the line $H_6 \cap H_7$. Among $\{H_i \cap \mathcal{P}\}_{1 \leq i \leq 9}$, two lines $H_6 \cap \mathcal{P}$, $H_7 \cap \mathcal{P}$ coincide, and dropping the line $H_1 \cap \mathcal{P}$, by the generic condition, it remains seven lines $\{H_i \cap \mathcal{P}\}_{i \neq 1,7}$ in general position in \mathcal{P} .



Letting B be the intersection point of the line $l = H_4 \cap H_5$ with the plane \mathcal{P} , the curve f lands in $\mathcal{P} \setminus (\cup_{1 \leq i \leq 9} H_i \cap \mathcal{P}) \setminus \{A, B\}$. As in (3.9), $f(\mathbb{C})$ is contained in some line, which is a contradiction.

Next, consider case (a2).



If \mathcal{P} contains some point in $\cup_{6 \leq i_1 < i_2 < i_3 \leq 9} H_{i_1} \cap H_{i_2} \cap H_{i_3}$, say $H_6 \cap H_7 \cap H_8$, then the curve f avoids three lines $H_i \cap \mathcal{P}$ ($6 \leq i \leq 8$), which are not in general position. By Theorem 2.6, $f(\mathbb{C})$ must be contained in some line, which is a contradiction. Therefore, \mathcal{P} does not contain any point in $\cup_{6 \leq i_1 < i_2 < i_3 \leq 9} H_{i_1} \cap H_{i_2} \cap H_{i_3}$. But then the curve f avoids a collection of four lines $\{H_i \cap \mathcal{P}\}_{6 \leq i \leq 9}$, which are in general position. By Theorem 2.3, its image must land in some diagonal line of this family, which is a contradiction.

Still in case (a), we can therefore assume that f is linearly nondegenerate. Assume that the omitted planes H_6, H_7, H_8, H_9 are given in the homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ by equations $\{z_i = 0\}$ ($0 \leq i \leq 3$). By Theorems 2.2, f has a reduced representation of the form

$$[1 : e^{\lambda_1 z + \mu_1} : e^{\lambda_2 z + \mu_2} : e^{\lambda_3 z + \mu_3}], \quad (3.10)$$

where λ_i, μ_i are constants with $\lambda_i \neq 0$ ($1 \leq i \leq 3$ and $\lambda_i \neq \lambda_j$ ($i \neq j$)). Let \mathcal{D} be the diagonal plane passing through the point $A = H_1 \cap H_2 \cap H_3$ and containing the line $l = H_4 \cap H_5$. By similar arguments as in Lemma 3.1, cf. (3.7), (3.8), we can show that

$$N_f(r, A) + N_f(r, l^*) \leq N_f(r, \mathcal{D}). \quad (3.11)$$

From the elementary inequality

$$\min \{\text{ord}_z(h_4 \circ f), 3\} + \min \{\text{ord}_z(h_5 \circ f), 3\} \leq 4 \min_{4 \leq i \leq 5} \text{ord}_z(h_i \circ f) \quad (z \in f^{-1}(l^*)),$$

by taking the sum on disks and then by integrating, we get

$$N_f^{[3]}(r, H_4) + N_f^{[3]}(r, H_5) \leq 4 N_f(r, l^*). \quad (3.12)$$

Next, we try to bound $N_f^{[3]}(r, H_i)$ ($1 \leq i \leq 3$) from above in terms of $N_f(r, A)$. Since f is of the form (3.10), for any $z_1, z_2 \in f^{-1}(A)$, we have

$$f^{(k)}(z_1) = f^{(k)}(z_2) \quad (k \in \mathbb{N}),$$

hence

$$\text{ord}_{z_1}(h_i \circ f) = \text{ord}_{z_2}(h_i \circ f) \quad (1 \leq i \leq 3). \quad (3.13)$$

Thus, it suffices to consider the two cases:

(a3) $\text{ord}_z(h_i \circ f) \leq 2$ for all $1 \leq i \leq 3$ and for all $z \in f^{-1}(A)$;

(a4) $\text{ord}_z(h_i \circ f) \geq 3$ for some i with $1 \leq i \leq 3$ and for all $z \in f^{-1}(A)$.

In case **(a3)**, the elementary inequality

$$\sum_{i=1}^3 \min \{ \text{ord}_z(h_i \circ f), 3 \} \leq 5 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \quad (z \in f^{-1}(A)),$$

yields

$$N_f^{[3]}(r, H_1) + N_f^{[3]}(r, H_2) + N_f^{[3]}(r, H_3) \leq 5 N_f(r, A). \quad (3.14)$$

Since f is linearly nondegenerate, we can proceed similarly as in (3.9)

$$\begin{aligned} 5 T_f(r) &\leq \sum_{i=1}^9 N_f^{[3]}(r, H_i) + S_f(r) \\ &\leq 5 N_f(r, A) + 4 N_f(r, l^*) + S_f(r) \\ &= 5 (N_f(r, A) + N_f(r, l^*)) - N_f(r, l^*) + S_f(r) \\ &\leq 5 N_f(r, \mathcal{D}) - N_f(r, l^*) + S_f(r) \\ &\leq 5 T_f(r) - N_f(r, l^*) + S_f(r). \end{aligned} \quad (3.15)$$

This implies

$$N_f(r, l^*) = S_f(r)$$

and hence, by (3.12), we have

$$N_f^{[3]}(r, H_4) + N_f^{[3]}(r, H_5) = S_f(r).$$

Therefore, the first inequality of (3.15) can be rewritten as

$$5 T_f(r) \leq \sum_{i=1}^3 N_f^{[3]}(r, H_i) + S_f(r).$$

By the First Main Theorem, the right-hand side of the above inequality is bounded from above by $3 T_f(r) + S_f(r)$. Thus we get

$$5 T_f(r) \leq 3 T_f(r) + S_f(r),$$

which is absurd.

Next, we consider case **(a4)**. Assume that $\text{ord}_z(h_1 \circ f) \geq 3$ for all $z \in f^{-1}(A)$. Since f is of the form (3.10), we claim that

$$\text{ord}_z(h_i \circ f) \leq 2 \quad (z \in f^{-1}(A), \quad 2 \leq i \leq 3). \quad (3.16)$$

Indeed, if $\text{ord}_z(h_i \circ f) \geq 3$ for some $z \in f^{-1}(A)$ and for some $2 \leq i \leq 3$, say $i = 2$, then $(e^{\lambda_1 z + \mu_1}, e^{\lambda_2 z + \mu_2}, e^{\lambda_3 z + \mu_3})$ is a solution of a system of six linear equations of the form

$$\begin{cases} 0 = a_{10} + a_{11} u + a_{12} v + a_{13} w, \\ 0 = a_{11} \lambda_1 u + a_{12} \lambda_2 v + a_{13} \lambda_3 w, \\ 0 = a_{11} \lambda_1^2 u + a_{12} \lambda_2^2 v + a_{13} \lambda_3^2 w, \\ 0 = a_{20} + a_{21} u + a_{22} v + a_{23} w, \\ 0 = a_{21} \lambda_1 u + a_{22} \lambda_2 v + a_{23} \lambda_3 w, \\ 0 = a_{21} \lambda_1^2 u + a_{22} \lambda_2^2 v + a_{23} \lambda_3^2 w, \end{cases}$$

where u, v, w are unknowns, and where a_{ij} ($0 \leq i \leq 3$) are the coefficients of h_i ($1 \leq i \leq 2$) in the homogeneous coordinate $[z_0 : z_1 : z_2 : z_3]$. Since λ_i are nonzero distinct constants, this forces the two linear forms h_1, h_2 to be linearly dependent, which is a contradiction.

It follows from (3.16) that

$$\min \{\text{ord}_z(h_2 \circ f), 3\} + \min \{\text{ord}_z(h_3 \circ f), 3\} \leq 3 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \quad (z \in f^{-1}(A)).$$

By taking the sum on disks and by integrating, we get

$$N_f^{[3]}(r, H_2) + N_f^{[3]}(r, H_3) \leq 3 N_f(r, A). \quad (3.17)$$

We may therefore proceed similarly as in (3.15)

$$\begin{aligned} 5 T_f(r) &\leq \sum_{i=1}^9 N_f^{[3]}(r, H_i) + S_f(r) \\ &\leq N_f^{[3]}(r, H_1) + 3 N_f(r, A) + 4 N_f(r, l^*) + S_f(r) \\ &\leq N_f(r, H_1) + 4 (N_f(r, A) + N_f(r, l^*)) - N_f(r, A) + S_f(r) \\ &\leq T_f(r) + 4 N_f(r, \mathcal{D}) - N_f(r, A) + S_f(r) \\ &\leq 5 T_f(r) - N_f(r, A) + S_f(r). \end{aligned} \quad (3.18)$$

This implies

$$N_f(r, A) = S_f(r).$$

By (3.17), we have

$$N_f^{[3]}(r, H_2) + N_f^{[3]}(r, H_3) = S_f(r).$$

Hence we can rewrite the first inequality of (3.18) and use First Main Theorem to get a contradiction

$$\begin{aligned} 5 T_f(r) &\leq N_f^{[3]}(r, H_1) + N_f^{[3]}(r, H_4) + N_f^{[3]}(r, H_5) + S_f(r) \\ &\leq 3 T_f(r) + S_f(r). \end{aligned}$$

Let us consider case **(b)**. Assume now $A_{2,3} = \{A, B\}$, where A, B are two points contained in $\cup_{1 \leq i_1 < i_2 < i_3 \leq 9} H_{i_1} \cap H_{i_2} \cap H_{i_3}$. There are three possibilities for the positions of A and B :

(b1) both A and B are contained in some line $H_i \cap H_j$;

(b2) both A and B are contained in some plane H_i but they are not contained in any line $H_i \cap H_j$;

(b3) there is no plane H_i containing both points A and B .

In case **(b1)**, the curve f avoids a family of five planes and, therefore, its image is contained in some diagonal plane of this family, which contains neither A nor B by the generic condition. Hence f avoids all planes H_i , which is absurd by Theorem 2.5.

Next, we consider case **(b2)**. Assume that $A = H_1 \cap H_2 \cap H_3$ and $B = H_1 \cap H_4 \cap H_5$, hence f avoids the 4 planes H_i ($6 \leq i \leq 9$). Similarly as in case **(a)**, the generic condition allows us to assume that f is linearly nondegenerate.

Since f avoids four planes, it is of the form (3.10) in some affine coordinates on $\mathbb{P}^3(\mathbb{C})$. Since f has no singular point, we have

$$\begin{aligned} \min_{i \in \{1,2,3\}} \text{ord}_z(h_i \circ f) &= 1 \quad (z \in f^{-1}(A)), \\ \min_{i \in \{1,4,5\}} \text{ord}_z(h_i \circ f) &= 1 \quad (z \in f^{-1}(B)). \end{aligned} \quad (3.19)$$

Hence by using these two equalities together with (3.16),

$$\sum_{i \in \{1,2,3\}} \min \{\text{ord}_z(h_i \circ f), 3\} \leq 6 = 6 \min_{i \in \{1,2,3\}} \text{ord}_z(h_i \circ f), \quad (z \in f^{-1}(A)),$$

$$\sum_{i \in \{1,4,5\}} \min \{\text{ord}_z(h_i \circ f), 3\} \leq 6 = 6 \min_{i \in \{1,4,5\}} \text{ord}_z(h_i \circ f), \quad (z \in f^{-1}(B)).$$

Thus, by taking the sum on disks of both sides of these inequalities and by integrating,

$$\sum_{i=1}^5 N_f^{[3]}(r, H_i) \leq 6 (N_f(r, A) + N_f(r, B)).$$

Next, using again that f is of the form (3.10), one can find two planes $\mathcal{P}_1 = \{\mathbf{p}_1 = 0\}$, $\mathcal{P}_2 = \{\mathbf{p}_2 = 0\}$ containing the line AB such that

$$\text{ord}_z(\mathbf{p}_1 \circ f) \geq 2 \quad (z \in f^{-1}(A)),$$

$$\text{ord}_z(\mathbf{p}_2 \circ f) \geq 2 \quad (z \in f^{-1}(B)).$$

Let $\mathcal{Q} = \{\mathbf{q} = \mathbf{p}_1 \mathbf{p}_2 = 0\}$ be the degenerate quadric $\mathcal{P}_1 \cup \mathcal{P}_2$. We have

$$3 = 3 \min_{i \in \{1,2,3\}} \text{ord}_z(h_i \circ f) \leq \text{ord}_z(\mathbf{p}_1 \circ f) + \text{ord}_z(\mathbf{p}_2 \circ f) = \text{ord}_z(\mathbf{q} \circ f) \quad (z \in f^{-1}(A)),$$

$$3 = 3 \min_{i \in \{1,4,5\}} \text{ord}_z(h_i \circ f) \leq \text{ord}_z(\mathbf{p}_1 \circ f) + \text{ord}_z(\mathbf{p}_2 \circ f) = \text{ord}_z(\mathbf{q} \circ f) \quad (z \in f^{-1}(B)),$$

which implies, by integrating, that

$$3 (N_f(r, A) + N_f(r, B)) \leq N_f(r, \mathcal{T}).$$

We proceed similarly as above to get a contradiction

$$\begin{aligned} 5 T_f(r) &\leq \sum_{i=1}^9 N_f^{[3]}(r, H_i) + S_f(r) \\ &\leq 6 (N_f(r, A) + N_f(r, B)) + S_f(r) \\ &\leq 2 N_f(r, \mathcal{T}) + S_f(r) \\ &\leq 4 T_f(r) + S_f(r). \end{aligned}$$

Now, we consider case **(b3)**. Assume that $A = H_1 \cap H_2 \cap H_3$, $B = H_4 \cap H_5 \cap H_6$, when f avoids the three planes H_7, H_8, H_9 . If $f(\mathbb{C})$ is contained in some plane \mathcal{P} , then it is not hard to see that \mathcal{P} must pass through both A and B . Furthermore, by using Theorem 2.6, one can show that \mathcal{P} does not pass through the point $C = H_7 \cap H_8 \cap H_9$. One can then always find 7 lines in general position in \mathcal{P} among $\{H_i \cap \mathcal{P}\}_{1 \leq i \leq 9}$. Hence one can use similar arguments as in Lemma 3.1, case $m = 2$, to get a contradiction. Thus, we can suppose that f is linearly nondegenerate.

Assume that the omitted planes H_7, H_8, H_9 are given in the homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ by the equations $\{z_0 = 0\}$, $\{z_1 = 0\}$, $\{z_2 = 0\}$. Since $\{H_i\}_{1 \leq i \leq 9}$ is a family of planes in general position, the planes H_i ($1 \leq i \leq 6$) are given by

$$h_i = \sum_{j=0}^3 a_{ij} z_j = 0,$$

with $a_{i3} \neq 0$ ($1 \leq i \leq 6$). Set $l_{i_1, i_2} = H_{i_1} \cap H_{i_2}$ ($1 \leq i_1 < i_2 \leq 3$), $l_{j_1, j_2} = H_{j_1} \cap H_{j_2}$ ($4 \leq j_1 < j_2 \leq 6$). For $1 \leq i < j \leq 3$ or $4 \leq i < j \leq 6$, let $R_{i,j} = \{r_{i,j} = 0\}$ be the plane containing the lines $AB, l_{i,j}$ and let $T_{i,j} = \{t_{i,j} = a_{j3} h_i - a_{i3} h_j = 0\}$ be the plane passing through the point $C = [0 : 0 : 0 : 1]$ and containing the line $l_{i,j}$. We note that all $r_{i,j}, t_{i,j}$ are linear combinations of h_i and h_j with nonzero coefficients.

Since f avoids three planes, by Theorem 2.2 it has a reduced representation of the form

$$[1 : e^{\lambda_1 z + \mu_1} : e^{\lambda_2 z + \mu_2} : g], \quad (3.20)$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are constants with $\lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \neq 0$ and where g is an entire function. Since f has no singular point, we have

$$\begin{aligned} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) &= 1 & (z \in f^{-1}(A)), \\ \min_{4 \leq j \leq 6} \text{ord}_z(h_j \circ f) &= 1 & (z \in f^{-1}(B)). \end{aligned} \quad (3.21)$$

Since f is of the form (3.20), we claim that

$$\begin{aligned} \min \{ \text{ord}_z(h_{i_1} \circ f), \text{ord}_z(h_{i_2} \circ f) \} &\leq 2 & (z \in f^{-1}(A), 1 \leq i_1 < i_2 \leq 3), \\ \min \{ \text{ord}_z(h_{j_1} \circ f), \text{ord}_z(h_{j_2} \circ f) \} &\leq 2 & (z \in f^{-1}(B), 4 \leq j_1 < j_2 \leq 6). \end{aligned} \quad (3.22)$$

Indeed, if one of these inequalities does not hold, say $\min \{ \text{ord}_z(h_1 \circ f), \text{ord}_z(h_2 \circ f) \} \geq 3$ for some $z \in f^{-1}(A)$, then z is a solution of the following system of equations

$$\begin{cases} 0 = (t_{1,2} \circ f)(z), \\ 0 = (t_{1,2} \circ f)'(z), \\ 0 = (t_{1,2} \circ f)''(z). \end{cases}$$

Equivalently, $(e^{\lambda_1 z + \mu_1}, e^{\lambda_2 z + \mu_2})$ is a solution of a system of three linear equations of the form

$$\begin{cases} 0 = (a_{23} a_{10} - a_{13} a_{20}) + (a_{23} a_{11} - a_{13} a_{21}) x + (a_{23} a_{12} - a_{13} a_{22}) y, \\ 0 = (a_{23} a_{11} - a_{13} a_{21}) \lambda_1 x + (a_{23} a_{12} - a_{13} a_{22}) \lambda_2 y, \\ 0 = (a_{23} a_{11} - a_{13} a_{21}) \lambda_1^2 x + (a_{23} a_{12} - a_{13} a_{22}) \lambda_2^2 y, \end{cases}$$

where x, y are unknowns. Since $\lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \neq 0$, this implies that the two linear forms h_1, h_2 must be linearly dependent, which is a contradiction.

It follows from (3.21) and (3.22) that

$$\begin{aligned} \sum_{i=1}^3 \min \{ \text{ord}_z(h_i \circ f), 3 \} &\leq 6 & (z \in f^{-1}(A)), \\ \sum_{j=4}^6 \min \{ \text{ord}_z(h_j \circ f), 3 \} &\leq 6 & (z \in f^{-1}(B)). \end{aligned} \quad (3.23)$$

Now we prove the following equality

Claim 3.1.

$$T_f(r) = N_f(r, A) + N_f(r, B) + S_f(r). \quad (3.24)$$

Proof. Since f is of the form (3.20) and since $t_{i,j}$ does not contain the term x_3 , we have

$$\begin{aligned} \text{ord}_{z_1}(t_{i_1, i_2} \circ f) &= \text{ord}_{z_2}(t_{i_1, i_2} \circ f) & (z_1, z_2 \in f^{-1}(A), 1 \leq i_1 < i_2 \leq 3), \\ \text{ord}_{z_1}(t_{j_1, j_2} \circ f) &= \text{ord}_{z_2}(t_{j_1, j_2} \circ f) & (z_1, z_2 \in f^{-1}(B), 4 \leq j_1 < j_2 \leq 6). \end{aligned} \quad (3.25)$$

Thus, it suffices to consider the four cases depending on f and $t_{i,j}$:

- (b3.1) $\text{ord}_z(t_{i_1, i_2} \circ f) = 1$ for all $1 \leq i_1 < i_2 \leq 3$, for all $z \in f^{-1}(A)$ and $\text{ord}_z(t_{j_1, j_2} \circ f) = 1$ for all $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$;
- (b3.2) $\text{ord}_z(t_{i_1, i_2} \circ f) \geq 2$ for some $1 \leq i_1 < i_2 \leq 3$, for all $z \in f^{-1}(A)$ and $\text{ord}_z(t_{j_1, j_2} \circ f) = 1$ for all $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$;
- (b3.3) $\text{ord}_z(t_{i_1, i_2} \circ f) = 1$ for all $1 \leq i_1 < i_2 \leq 3$, for all $z \in f^{-1}(A)$ and $\text{ord}_z(t_{j_1, j_2} \circ f) \geq 2$ for some $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$;

(b3.4) $\text{ord}_z(t_{i_1, i_2} \circ f) \geq 2$ for some $1 \leq i_1 < i_2 \leq 3$, for all $z \in f^{-1}(A)$ and $\text{ord}_z(t_{j_1, j_2} \circ f) \geq 2$ for some $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$.

Consider case **(b3.1)**. Since $t_{i,j}$ is a linear combination of h_i and h_j with nonzero coefficients, we have

$$\begin{aligned} \min \{\text{ord}_z(h_{i_1} \circ f), \text{ord}_z(h_{i_2} \circ f)\} &= 1 & (z \in f^{-1}(A) \quad 1 \leq i_1 < i_2 \leq 3), \\ \min \{\text{ord}_z(h_{j_1} \circ f), \text{ord}_z(h_{j_2} \circ f)\} &= 1 & (z \in f^{-1}(B), \quad 4 \leq j_1 < j_2 \leq 6). \end{aligned}$$

Using these equalities together with (3.21), we get

$$\begin{aligned} \sum_{i=1}^3 \min \{\text{ord}_z(h_i \circ f), 3\} &\leq 5 = 5 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) & (z \in f^{-1}(A)), \\ \sum_{i=4}^6 \min \{\text{ord}_z(h_i \circ f), 3\} &\leq 5 = 5 \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f) & (z \in f^{-1}(B)). \end{aligned} \quad (3.26)$$

By taking the sum on disks and by integrating these two inequalities, we obtain

$$\begin{aligned} N_f^{[3]}(r, H_1) + N_f^{[3]}(r, H_2) + N_f^{[3]}(r, H_3) &\leq 5 N_f(r, A), \\ N_f^{[3]}(r, H_4) + N_f^{[3]}(r, H_5) + N_f^{[3]}(r, H_6) &\leq 5 N_f(r, B). \end{aligned}$$

Letting \mathcal{B} be a plane passing through A and B , we proceed similarly as before

$$\begin{aligned} 5 T_f(r) &\leq \sum_{i=1}^9 N_f^{[3]}(r, H_i) + S_f(r) \\ &\leq 5 N_f(r, A) + 5 N_f(r, B) + S_f(r) \\ &\leq 5 N_f(r, \mathcal{B}) + S_f(r) \\ &\leq 5 T_f(r) + S_f(r). \end{aligned} \quad (3.27)$$

Here, $S_f(r) = o(T_f(r))$ is negligible, hence all inequalities are equalities modulo $S_f(r)$. This gives (3.24), as wanted.

Next, we consider case **(b3.2)**. Let us set

$$\begin{aligned} E_{t,A} &= \{z \in \mathbb{C} : |z| < t, f(z) = A\}, \\ E_{t,A,i}^1 &= \{z \in \mathbb{C} : |z| < t, f(z) = A, \text{ord}_z(h_i \circ f) = 1\} & (1 \leq i \leq 3), \\ E_{t,A,i}^{\geq 2} &= \{z \in \mathbb{C} : |z| < t, f(z) = A, \text{ord}_z(h_i \circ f) \geq 2\} & (1 \leq i \leq 3), \\ E_{t,B} &= \{z \in \mathbb{C} : |z| < t, f(z) = B\}, \\ E_{t,B,i}^1 &= \{z \in \mathbb{C} : |z| < t, f(z) = B, \text{ord}_z(h_i \circ f) = 1\} & (4 \leq i \leq 6), \\ E_{t,B,i}^{\geq 2} &= \{z \in \mathbb{C} : |z| < t, f(z) = B, \text{ord}_z(h_i \circ f) \geq 2\} & (4 \leq i \leq 6). \end{aligned}$$

Assume that $\text{ord}_z(t_{1,2} \circ f) \geq 2$ for all $z \in f^{-1}(A)$. Since $t_{1,2}$, $r_{1,2}$ are linear combinations of h_1 and h_2 with nonzero coefficients, we have

$$\begin{aligned} E_{t,A,1}^{\geq 2} &= E_{t,A,2}^{\geq 2}, \\ \text{ord}_z(r_{1,2} \circ f) &\geq 2 & (z \in E_{t,A,1}^{\geq 2}). \end{aligned} \quad (3.28)$$

For the same reason

$$E_{t,A,1}^1 = E_{t,A,2}^1,$$

which yields

$$\sum_{i=1}^3 \min \{\text{ord}_z(h_i \circ f), 3\} \leq 5 \quad (z \in E_{t,A,1}^1). \quad (3.29)$$

Letting $\mathcal{R} = \{r = r_{1,2} r_{4,5} r_{5,6} r_{4,6} = 0\}$ be the degenerate quartic $R_{1,2} \cup R_{4,5} \cup R_{5,6} \cup R_{4,6}$ whose four components pass through A and B , we have

$$\text{ord}_z(\mathbf{r} \circ f) \geq 4 \quad (z \in E_{t,A} \cup E_{t,B}). \quad (3.30)$$

Furthermore, it follows from (3.28) that

$$\text{ord}_z(\mathbf{r} \circ f) \geq 5 \quad (z \in E_{t,A,1}^{\geq 2}).$$

Using this inequality together with (3.23) and (3.21), we get

$$\sum_{i=1}^3 \min \{\text{ord}_z(h_i \circ f), 3\} \leq 6 = 6 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \leq \frac{6}{5} \text{ord}_z(\mathbf{r} \circ f) \quad (z \in E_{t,A,1}^{\geq 2}).$$

Combining (3.29), (3.21) and (3.30), we receive

$$\sum_{i=1}^3 \min \{\text{ord}_z(h_i \circ f), 3\} \leq 5 = 5 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \leq \frac{5}{4} \text{ord}_z(\mathbf{r} \circ f) \quad (z \in E_{t,A,1}^1).$$

Since $\text{ord}_z(t_{j_1, j_2} \circ f) = 1$ for all $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$, by similar arguments as in (3.26) and by using (3.30), we also have

$$\sum_{i=4}^6 \min \{\text{ord}_z(h_i \circ f), 3\} \leq 5 = 5 \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f) \leq \frac{5}{4} \text{ord}_z(\mathbf{r} \circ f) \quad (z \in E_{t,B}).$$

By taking the sum on disks and by integrating these three inequalities, we obtain

$$\begin{aligned} \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt &\leq 6 \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt \\ &\leq \frac{6}{5} \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(\mathbf{r} \circ f)}{t} dt, \end{aligned} \quad (3.31)$$

$$\begin{aligned} \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt &\leq 5 \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt \\ &\leq \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \text{ord}_z(\mathbf{r} \circ f)}{t} dt, \end{aligned} \quad (3.32)$$

$$\begin{aligned} \sum_{i=4}^6 \int_1^r \frac{\sum_{z \in E_{t,B}} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt &\leq 5 N_f(r, B) \\ &\leq \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{t,B}} \text{ord}_z(\mathbf{r} \circ f)}{t} dt. \end{aligned} \quad (3.33)$$

We then proceed similarly as before:

$$\begin{aligned}
5T_f(r) &\leq \sum_{i=1}^9 N_f^{[3]}(r, H_i) + S_f(r) \\
&\leq \sum_{i=1}^3 N_f^{[3]}(r, H_i) + \sum_{i=4}^6 N_f^{[3]}(r, H_i) + S_f(r) \\
&= \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A}} \min\{\text{ord}_z(h_i \circ f), 3\}}{t} dt + \sum_{i=4}^6 \int_1^r \frac{\sum_{z \in E_{t,B}} \min\{\text{ord}_z(h_i \circ f), 3\}}{t} dt + S_f(r) \\
&= \sum_{i=1}^3 \left(\int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min\{\text{ord}_z(h_i \circ f), 3\}}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min\{\text{ord}_z(h_i \circ f), 3\}}{t} dt \right) \\
&\quad + \sum_{i=4}^6 \int_1^r \frac{\sum_{z \in E_{t,B}} \min\{\text{ord}_z(h_i \circ f), 3\}}{t} dt + S_f(r) \\
&\leq \frac{6}{5} \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(r \circ f)}{t} dt + \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \text{ord}_z(r \circ f)}{t} dt \\
&\quad + \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{t,B}} \text{ord}_z(r \circ f)}{t} dt + S_f(r) \\
&= \frac{5}{4} \left(\int_1^r \frac{\sum_{z \in E_{t,A}} \text{ord}_z(r \circ f)}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,B}} \text{ord}_z(r \circ f)}{t} dt \right) \\
&\quad + \left(\frac{6}{5} - \frac{5}{4} \right) \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(r \circ f)}{t} dt + S_f(r) \\
&\leq \frac{5}{4} N_f(r, \mathcal{R}) - \frac{1}{20} \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(r \circ f)}{t} dt + S_f(r) \\
&\leq 5T_f(r) - \frac{1}{20} \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(r \circ f)}{t} dt + S_f(r). \tag{3.34}
\end{aligned}$$

This implies

$$\int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(r \circ f)}{t} dt = S_f(r) \tag{3.35}$$

and whence all inequalities in (3.34) become equalities modulo $S_f(r)$, which gives

$$\sum_{i=1}^6 N_f^{[3]}(r, H_i) = 5T_f(r) + S_f(r), \tag{3.36}$$

$$\sum_{i=1}^3 N_f^{[3]}(r, H_i) = \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{t,A}} \text{ord}_z(r \circ f)}{t} dt + S_f(r), \tag{3.37}$$

$$\sum_{i=4}^6 N_f^{[3]}(r, H_i) = \frac{5}{4} \int_1^r \frac{\sum_{z \in E_{t,B}} \text{ord}_z(r \circ f)}{t} dt + S_f(r). \tag{3.38}$$

It follows from (3.33) and (3.38) that

$$\sum_{i=4}^6 N_f^{[3]}(r, H_i) = 5N_f(r, B) + S_f(r). \tag{3.39}$$

Owing to (3.35), the two inequalities (3.31) become

$$\sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min\{\text{ord}_z(h_i \circ f), 3\}}{t} dt = S_f(r)$$

$$\int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt = S_f(r).$$

Hence

$$\sum_{i=1}^3 N_f^{[3]}(r, H_i) = \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt + S_f(r), \quad (3.40)$$

$$N_f(r, A) = \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt + S_f(r). \quad (3.41)$$

Combining (3.32), (3.40), (3.41), we get

$$\sum_{i=1}^3 N_f^{[3]}(r, H_i) = 5 N_f(r, A) + S_f(r). \quad (3.42)$$

The equality (3.24) follows from (3.36), (3.39), (3.42).

Case **(b3.3)** can be treated by similar arguments as for case **(b3.2)**.

Next, we consider case **(b3.4)**. Assume that

$$\text{ord}_z(t_{1,2} \circ f) \geq 2 \quad (z \in f^{-1}(A)),$$

$$\text{ord}_z(t_{4,5} \circ f) \geq 2 \quad (z \in f^{-1}(B)).$$

By similar argument as in (3.28), we have $E_{t,A,1}^{\geq 2} = E_{t,A,2}^{\geq 2}$, $E_{t,B,4}^{\geq 2} = E_{t,B,5}^{\geq 2}$, $E_{t,A,1}^1 = E_{t,A,2}^1$, $E_{t,B,4}^1 = E_{t,B,5}^1$, which implies

$$\text{ord}_z(r_{1,2} \circ f) \geq 2 \quad (z \in E_{t,A,1}^{\geq 2}), \quad (3.43)$$

$$\text{ord}_z(r_{4,5} \circ f) \geq 2 \quad (z \in E_{t,B,4}^{\geq 2}), \quad (3.44)$$

$$\begin{aligned} \sum_{i=1}^3 \min \{\text{ord}_z(h_i \circ f), 3\} &\leq 5 \quad (z \in E_{t,A,1}^1), \\ \sum_{i=4}^6 \min \{\text{ord}_z(h_i \circ f), 3\} &\leq 5 \quad (z \in E_{t,B,4}^1). \end{aligned}$$

Letting $\mathcal{S} = \{s = r_{1,2} r_{4,5} = 0\}$ be the degenerate quadric $R_{1,2} \cup R_{4,5}$, we see that

$$\text{ord}_z(s \circ f) = \text{ord}_z(r_{1,2} \circ f) + \text{ord}_z(r_{4,5} \circ f) \geq 2 \quad (z \in E_{t,A} \cup E_{t,B}).$$

Furthermore, by using (3.43) and (3.44), we have

$$\text{ord}_z(s \circ f) = \text{ord}_z(r_{1,2} \circ f) + \text{ord}_z(r_{4,5} \circ f) \geq 3 \quad (z \in E_{t,A,1}^{\geq 2} \cup E_{t,B,4}^{\geq 2}).$$

Similarly as in the previous case, by using these inequalities together with (3.21) and (3.23), we receive

$$\begin{aligned} \sum_{i=1}^3 \min \{\text{ord}_z(h_i \circ f), 3\} &\leq 6 = 6 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \leq \frac{6}{3} \text{ord}_z(s \circ f) \quad (z \in E_{t,A,1}^{\geq 2}), \\ \sum_{i=1}^3 \min \{\text{ord}_z(h_i \circ f), 3\} &\leq 5 = 5 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \leq \frac{5}{2} \text{ord}_z(s \circ f) \quad (z \in E_{t,A,1}^1), \\ \sum_{i=4}^6 \min \{\text{ord}_z(h_i \circ f), 3\} &\leq 6 = 6 \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f) \leq \frac{6}{3} \text{ord}_z(s \circ f) \quad (z \in E_{t,B,4}^{\geq 2}), \\ \sum_{i=4}^6 \min \{\text{ord}_z(h_i \circ f), 3\} &\leq 5 = 5 \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f) \leq \frac{5}{2} \text{ord}_z(s \circ f) \quad (z \in E_{t,B,4}^1). \end{aligned}$$

By taking the sum on disks and by integrating these four inequalities, we obtain

$$\begin{aligned} \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt &\leq 6 \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt \\ &\leq 2 \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt, \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt &\leq 5 \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt \\ &\leq \frac{5}{2} \int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \text{ord}_z(s \circ f)}{t} dt, \end{aligned}$$

$$\begin{aligned} \sum_{i=4}^6 \int_1^r \frac{\sum_{z \in E_{t,B,4}^{\geq 2}} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt &\leq 6 \int_1^r \frac{\sum_{z \in E_{t,B,4}^{\geq 2}} \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f)}{t} dt \\ &\leq 2 \int_1^r \frac{\sum_{z \in E_{t,B,4}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt, \end{aligned}$$

$$\begin{aligned} \sum_{i=4}^6 \int_1^r \frac{\sum_{z \in E_{t,B,4}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt &\leq 5 \int_1^r \frac{\sum_{z \in E_{t,B,4}^1} \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f)}{t} dt \\ &\leq \frac{5}{2} \int_1^r \frac{\sum_{z \in E_{t,B,4}^1} \text{ord}_z(s \circ f)}{t} dt. \end{aligned}$$

Now, we proceed similarly as above

$$\begin{aligned} 5T_f(r) &\leq \sum_{i=1}^9 N_f^{[3]}(r, H_i) + S_f(r) \\ &= \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A}} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt + \sum_{i=4}^6 \int_1^r \frac{\sum_{z \in E_{t,B}} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt + S_f(r) \\ &= \sum_{i=1}^3 \left(\int_1^r \frac{\sum_{z \in E_{t,A,1}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt \right) \\ &\quad + \sum_{i=4}^6 \left(\int_1^r \frac{\sum_{z \in E_{t,B,1}^1} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,B,4}^{\geq 2}} \min \{\text{ord}_z(h_i \circ f), 3\}}{t} dt \right) + S_f(r) \\ &\leq \frac{5}{2} \left(\int_1^r \frac{\sum_{z \in E_{t,A}} \text{ord}_z(s \circ f)}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,B}} \text{ord}_z(s \circ f)}{t} dt \right) \\ &\quad + \left(2 - \frac{5}{2} \right) \left(\int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,B,4}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt \right) + S_f(r) \\ &\leq \frac{5}{2} N_f(r, \mathcal{S}) - \frac{1}{2} \left(\int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,B,4}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt \right) + S_f(r) \\ &\leq 5T_f(r) - \frac{1}{2} \left(\int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,B,4}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt \right) + S_f(r). \end{aligned}$$

This implies

$$\begin{aligned}
\int_1^r \frac{\sum_{z \in E_{t,A,1}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt &= S_f(r), \\
\int_1^r \frac{\sum_{z \in E_{t,B,4}^{\geq 2}} \text{ord}_z(s \circ f)}{t} dt &= S_f(r), \\
\sum_{i=1}^6 N_f^{[3]}(r, H_i) &= 5T_f(r) + S_f(r), \\
\sum_{i=1}^3 N_f^{[3]}(r, H_i) &= \frac{5}{2} \int_1^r \frac{\sum_{z \in E_{t,A}} \text{ord}_z(r \circ f)}{t} dt + S_f(r), \\
\sum_{i=4}^6 N_f^{[3]}(r, H_i) &= \frac{5}{2} \int_1^r \frac{\sum_{z \in E_{t,B}} \text{ord}_z(r \circ f)}{t} dt + S_f(r).
\end{aligned}$$

By proceeding similarly as in (3.42), we receive

$$\begin{aligned}
\sum_{i=1}^3 N_f^{[3]}(r, H_i) &= 5N_f(r, A) + S_f(r), \\
\sum_{i=4}^6 N_f^{[3]}(r, H_i) &= 5N_f(r, B) + S_f(r).
\end{aligned}$$

Hence, the equality (3.24) also holds in this case. Claim 3.1 is thus proved. \square

Next, since f is of the form (3.20), one can find a plane $\mathcal{K} = \{\mathbf{k} = 0\}$ passing through A and C such that

$$\text{ord}_z(\mathbf{k} \circ f) \geq 2 \quad (z \in f^{-1}(A)).$$

Let $\mathcal{B}_i = \{\mathbf{b}_i = 0\}$ be the plane containing the two lines AB , $H_i \cap \mathcal{K}$ ($1 \leq i \leq 3$). Since \mathbf{b}_i is a linear combination of h_i and \mathbf{k} with nonzero coefficients, we have

$$\text{ord}_z(\mathbf{b}_i \circ f) \geq 2 \quad (z \in E_{t,A,i}^{\geq 2}),$$

which yields

$$\sum_{i=1}^3 \text{ord}_z(\mathbf{b}_i \circ f) \geq 4 \quad (z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}). \quad (3.45)$$

Let $\mathcal{C} = \{\mathbf{c} = \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 = 0\}$ be the degenerate cubic $\cup_{1 \leq i \leq 3} \mathcal{B}_i$. It follows from (3.21) and (3.45) that

$$\begin{aligned}
\min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) &= 1 \leq \frac{1}{4} \sum_{i=1}^3 \text{ord}_z(\mathbf{b}_i \circ f) = \frac{1}{4} \text{ord}_z(\mathbf{c} \circ f) \quad (z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}), \\
\min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) &= 1 \leq \frac{1}{3} \sum_{i=1}^3 \text{ord}_z(\mathbf{b}_i \circ f) = \frac{1}{3} \text{ord}_z(\mathbf{c} \circ f) \quad (z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}), \\
\min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f) &= 1 \leq \frac{1}{3} \sum_{i=1}^3 \text{ord}_z(\mathbf{b}_i \circ f) = \frac{1}{3} \text{ord}_z(\mathbf{c} \circ f) \quad (z \in E_{t,B}).
\end{aligned}$$

By taking the sum on disks and by integrating these inequalities,

$$\begin{aligned}
&\int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt \leq \frac{1}{4} \int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \text{ord}_z(\mathbf{c} \circ f)}{t} dt, \\
&\int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt \leq \frac{1}{3} \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \text{ord}_z(\mathbf{c} \circ f)}{t} dt, \\
N_f(r, B) &= \int_1^r \frac{\sum_{z \in E_{t,B}} \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f)}{t} dt \leq \frac{1}{3} \int_1^r \frac{\sum_{z \in E_{t,B}} \text{ord}_z(\mathbf{c} \circ f)}{t} dt.
\end{aligned}$$

By using these inequalities together with (3.24), we receive

$$\begin{aligned}
5T_f(r) &= 5N_f(r, A) + 5N_f(r, B) + S_f(r) \\
&= 5 \left(\int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt \right) \\
&\quad + 5 \int_1^r \frac{\sum_{z \in E_{t,B}} \min_{4 \leq i \leq 6} \text{ord}_z(h_i \circ f)}{t} dt + S_f(r) \\
&\leq \frac{5}{3} \left(\int_1^r \frac{\sum_{z \in E_{t,A}} \text{ord}_z(c \circ f)}{t} dt + \int_1^r \frac{\sum_{z \in E_{t,B}} \text{ord}_z(c \circ f)}{t} dt \right) \\
&\quad + \left(\frac{5}{4} - \frac{5}{3} \right) \int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \text{ord}_z(c \circ f)}{t} dt + S_f(r) \\
&\leq \frac{5}{3} N_f(r, C) - \frac{5}{12} \int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \text{ord}_z(c \circ f)}{t} dt + S_f(r) \\
&\leq 5T_f(r) - \frac{5}{12} \int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \text{ord}_z(c \circ f)}{t} dt + S_f(r).
\end{aligned}$$

This implies

$$\int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \text{ord}_z(c \circ f)}{t} dt = S_f(r). \quad (3.46)$$

By using (3.23) and (3.45), we get

$$\sum_{i=1}^3 \min \{ \text{ord}_z(h_i \circ f), 3 \} \leq 6 \leq \frac{3}{2} \text{ord}_z(c \circ f) \quad (z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}),$$

which yields

$$\begin{aligned}
\int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt &\leq \frac{3}{2} \int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \text{ord}_z(c \circ f)}{t} dt \\
&\stackrel{[\text{Use (3.46)}]}{=} S_f(r).
\end{aligned} \quad (3.47)$$

Moreover, we also have

$$\sum_{i=1}^3 \min \{ \text{ord}_z(h_i \circ f), 3 \} = 3 = 3 \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f) \quad (z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}),$$

which implies, by integrating, that

$$\sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min \{ \text{ord}_z(h_i \circ f), 3 \}}{t} dt \leq 3 \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt. \quad (3.48)$$

By combining (3.47) and (3.48), we get

$$\begin{aligned}
\sum_{i=1}^3 N_f^{[3]}(r, H_i) &= \sum_{i=1}^3 \int_1^r \frac{\sum_{z \in E_{t,A}} \min \{ \text{ord}_z(h_i \circ f), 3 \}}{t} dt \\
&= \sum_{i=1}^3 \left(\int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min \{ \text{ord}_z(h_i \circ f), 3 \}}{t} dt \right. \\
&\quad \left. + \int_1^r \frac{\sum_{z \in \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min \{ \text{ord}_z(h_i \circ f), 3 \}}{t} dt \right) + S_f(r) \\
&\leq 3 \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^3 E_{t,A,i}^{\geq 2}} \min_{1 \leq i \leq 3} \text{ord}_z(h_i \circ f)}{t} dt + S_f(r) \\
&\leq 3N_f(r, A) + S_f(r).
\end{aligned}$$

By symmetry, we also have

$$\sum_{i=4}^6 N_f^{[3]}(r, H_i) \leq 3N_f(r, B) + S_f(r).$$

Hence we can rewrite (3.27) to get a contradiction:

$$\begin{aligned} 5T_f(r) &\leq \sum_{i=1}^9 N_f^{[3]}(r, H_i) + S_f(r) \\ &\leq 3N_f(r, A) + 3N_f(r, B) + S_f(r) \\ &\leq 3N_f(r, \mathcal{B}) + S_f(r) \\ &\leq 3T_f(r) + S_f(r). \end{aligned}$$

□

In $\mathbb{P}^4(\mathbb{C})$, by the generic condition for the family of hyperplanes $\{H_i\}_{1 \leq i \leq q}$, when $q \geq 10$, we see that, for all three disjoint subsets I, J, K of $\{1, \dots, q\}$ with $|I| \geq 2$, $|J| \geq 2$, $|I| + |J| = 6$, $|K| = 4$, the diagonal hyperplane H_{IJ} does not contain the point $\cap_{k \in K} H_k$.

Lemma 3.3. *In $\mathbb{P}^4(\mathbb{C})$, all complements of the form (3.2) are hyperbolic if $m = 1$.*

Proof. We can assume that $A_{1,4}$ is a set consisting of one element in

$$\left(\bigcup_{1 \leq i_1 < i_2 \leq 10} (H_{i_1} \cap H_{i_2})^* \right) \bigcup \left(\bigcup_{1 \leq i_1 < i_2 < i_3 \leq 10} H_{i_1} \cap H_{i_2} \cap H_{i_3} \right)^* \bigcup \left(\bigcup_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 10} H_{i_1} \cap H_{i_2} \cap H_{i_3} \cap H_{i_4} \right).$$

Suppose to the contrary that there is an entire curve $f: \mathbb{C} \rightarrow \mathbb{P}^4(\mathbb{C}) \setminus \left(\bigcup_{i=1}^{10} H_i \setminus A_{1,4} \right)$. If $A_{1,4}$ is not a set of a point, then f avoids at least seven hyperplanes. By Theorem 2.4, its image is contained in a line L and we can continue to analyze the position of L with respect to $\bigcup_{i=1}^{10} H_i \setminus A_{1,4}$ to get a contradiction. Consider the remaining case where $A_{1,4}$ consists of a point, say $\cap_{i=1}^4 H_i$. By Theorem 2.3, the curve f lands in some diagonal hyperplane of the family $\{H_i\}_{5 \leq i \leq 10}$, which does not contain the point $\cap_{i=1}^4 H_i$ by the generic condition. Hence, f must avoid all H_i ($1 \leq i \leq 10$), which is impossible by Theorem 2.5. □

3.1 Stability of intersections

We will also invoke the following known complex analysis fact.

Stability of intersections. *Let X be a complex manifold and let $H \subset X$ be an analytic hypersurface. Suppose that a sequence (f_n) of entire curves in X converges toward an entire curve f . If $f(\mathbb{C})$ is not contained in H , then we have*

$$f(\mathbb{C}) \cap H \subset \lim f_n(\mathbb{C}) \cap H.$$

4 Proof of the Main Theorem

We keep the notation of the previous section. Let S be a hypersurface of degree $2n$, which is in general position with respect to the family $\{H_i\}_{1 \leq i \leq 2n}$. We would like to determine what conditions S should satisfy for Σ_ϵ to be hyperbolic. Suppose that Σ_{ϵ_k} is not hyperbolic for a sequence (ϵ_k) converging to 0. Then we can find entire curves $f_{\epsilon_k}: \mathbb{C} \rightarrow \Sigma_{\epsilon_k}$. By the Brody lemma, after reparameterization and extraction, we may assume that the sequence (f_{ϵ_k}) converges to an entire curve $f: \mathbb{C} \rightarrow \bigcup_{i=1}^{2n} H_i$. The curve $f(\mathbb{C})$ lands inside some hyperplane H_i . Moreover, it cannot land inside any subspace of dimension 1 (a line). Indeed, if $f(\mathbb{C}) \subset \cap_{i \in I} H_i$ for some subset I of the index set $\mathbf{Q} = \{1, \dots, 2n\}$ having cardinality $n - 1$, then for all $j \in \mathbf{Q} \setminus I$, by stability of intersections, one has

$$f(\mathbb{C}) \cap H_j \subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j \subset \lim \Sigma_{\epsilon_k} \cap H_j \subset S \cap H_j.$$

Thus $f(\mathbb{C})$ and H_j have empty intersection by the general position. Hence the curve $f(\mathbb{C})$ lands in

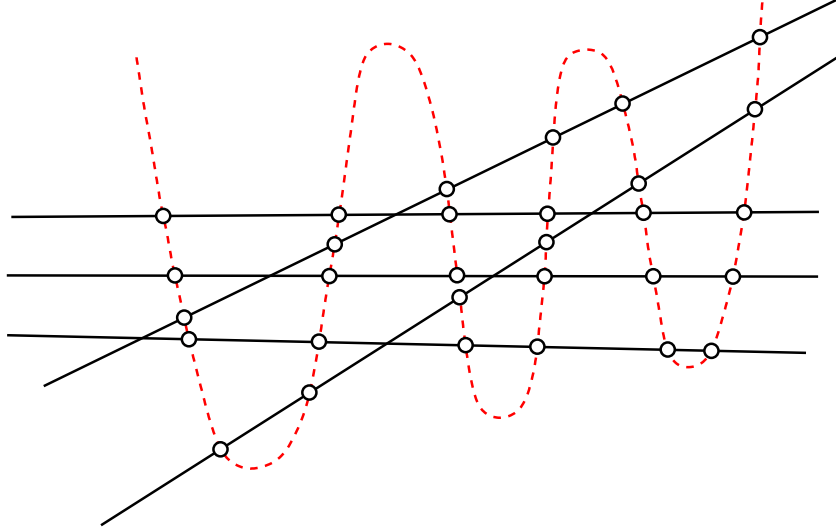
$$\cap_{i \in I} H_i \setminus \left(\bigcup_{j \in \mathbf{Q} \setminus I} H_j \right).$$

This is a contradiction, because the complement of $n + 1$ ($n \geq 3$) points in a line is hyperbolic by Picard's theorem.

Now, let I be the largest subset of \mathbf{Q} such that the curve $f(\mathbb{C})$ lands in $\cap_{i \in I} H_i$. We have $|I| \leq n - 2$. By stability of intersections, $f(\mathbb{C}) \cap H_j$ is contained in S for all $j \in \mathbf{Q} \setminus I$. Therefore the curve $f(\mathbb{C})$ lands in

$$\cap_{i \in I} H_i \setminus \left(\cup_{j \in \mathbf{Q} \setminus I} H_j \setminus S \right). \quad (4.1)$$

So, the problem reduces to finding a hypersurface S of degree $2n$ such that all complements of the form (4.1) are hyperbolic, where I is a subset of \mathbf{Q} of cardinality at most $n - 2$. For example when $n = 3$ ([10]), we need to find a sextic curve S such that all complements of the form $H_i \setminus \left(\cup_{j \neq i} H_j \setminus S \right)$ are hyperbolic. In this case, we have the complement of five lines in the hyperplane H_i on which all points of intersection with S are deleted.



We will construct such S by deformation, step by step. For $2 \leq l \leq n - 1$, let Δ_l be a finite collection of subspaces of dimension $n - l$, in the sense of section 3. Let $D_l \notin \Delta_l$ be another subspace of dimension $n - l$, defined as $D_l = \cap_{i \in I_{D_l}} H_i$. For a hypersurface $S = \{s = 0\}$ in general position with respect to the family $\{H_i\}_{1 \leq i \leq 2n}$ and $\epsilon \neq 0$, we set

$$S_\epsilon = \{\epsilon s + \prod_{i \notin I_{D_l}} h_i^{n_i} = 0\},$$

where $n_i \geq 1$ are chosen (freely) so that $\sum_{i \notin I_{D_l}} n_i = 2n$. It is not hard to see that the hypersurface S_ϵ is also in general position with respect to the family $\{H_i\}_{1 \leq i \leq 2n}$. We denote by $\overline{\Delta}_l$ the family of all subspaces of dimension $n - l$ ($2 \leq l \leq n$) with the convention $\overline{\Delta}_n = \emptyset$.

Lemma 4.1. *Assume that all complements of the form*

$$\cap_{i \in I} H_i \setminus \left(\cup_{j \in J} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m, n-|I|}) \right) \quad (4.2)$$

are hyperbolic where I, J are two disjoint subsets of $\{1, \dots, 2n\}$ such that $|I| \leq n - 2$, $|J| + 2|I| \geq 2n + 1$ and $m \leq |J| + 2|I| - (2n + 1)$. Here, $A_{m, n-|I|}$ is a set of at most m star-subspaces coming from the family of hyperplanes $\{\cap_{i \in I} H_i \cap H_j\}_{j \in J}$ in $\cap_{i \in I} H_i \cong \mathbb{P}^{n-|I|}(\mathbb{C})$. Then all complements of the form

$$\cap_{i \in I} H_i \setminus \left(\cup_{j \in J} H_j \setminus (((\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_\epsilon) \cup A_{m, n-|I|}) \right) \quad (4.3)$$

are also hyperbolic for sufficiently small $\epsilon \neq 0$.

Proof. By the definition of S_ϵ , we see that $S_\epsilon \cap (\cap_{m \in M} H_m) = S \cap (\cap_{m \in M} H_m)$ when $M \cap (\mathbf{Q} \setminus I_{D_l}) \neq \emptyset$, hence

$$(\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_\epsilon = ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup (D_l \cap S_\epsilon).$$

When $|I| \geq l$, using this, we observe that two complements (4.2), (4.3) coincide.

Assume therefore $|I| \leq l-1$. Suppose by contradiction that there exists a sequence of entire curves $(f_{\epsilon_k}(\mathbb{C}))_k$, $\epsilon_k \rightarrow 0$ contained in the complement

$$\cap_{i \in I} H_i \setminus \left(\cup_{j \in J} H_j \setminus (((\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_{\epsilon_k}) \cup A_{m,n-|I|}) \right).$$

By the Brody Lemma, we may assume that (f_{ϵ_k}) converges to an entire curve $f(\mathbb{C}) \subset \cap_{i \in I} H_i$. Our aim is to prove that the curve $f(\mathbb{C})$ lands in some complement of the form (4.2). Let $\cap_{k \in K} H_k$ be the smallest subspace containing $f(\mathbb{C})$. It is clear that $K \supset I$. Take an index j in $J \setminus K$. By stability of intersections, one has

$$\begin{aligned} f(\mathbb{C}) \cap H_j &\subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j \\ &\subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|} \cup \lim(D_l \cap S_{\epsilon_k}). \end{aligned} \quad (4.4)$$

If the index j does not belong to I_{D_l} , then $H_j \cap D_l \cap S_{\epsilon_k} \subset \overline{\Delta}_{l+1} \cap S$. It follows from (4.4) that

$$f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|}. \quad (4.5)$$

If the index j belongs to I_{D_l} , noting that $\lim(D_l \cap S_{\epsilon_k})$ is contained in $D_l \cap (\cup_{i \notin I_{D_l}} H_i)$, again from (4.4), one has

$$f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|} \cup (D_l \cap (\cup_{i \notin I_{D_l}} H_i)). \quad (4.6)$$

Assume first that $K = I$. We claim that (4.5) also holds when the index $j \in J \setminus I$ belonging to I_{D_l} . Indeed, for the supplementary part in (4.6), we have

$$f(\mathbb{C}) \cap H_j \cap (D_l \cup_{i \notin I_{D_l}} H_i) \subset \cup_{i \notin I_{D_l}} (f(\mathbb{C}) \cap H_j \cap H_i),$$

so that (4.5) applies here to all $i \notin I_{D_l}$. Hence, the curve $f(\mathbb{C})$ lands inside

$$\cap_{i \in I} H_i \setminus \left(\cup_{j \in J} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|}) \right),$$

contradicting the hypothesis.

Assume now that I is a proper subset of K . Let us set

$$A_{m,n-|I|,K} = \{X \cap (\cap_{k \in K} H_k) \mid X \in A_{m,n-|I|}\}.$$

This set consists of star-subspaces of $\cap_{k \in K} H_k \cong \mathbb{P}^{n-|K|}(\mathbb{C})$. Let $B_{m,K}$ be the subset of $A_{m,n-|I|,K}$ containing all star-subspaces of dimension $n - |K| - 1$ (i.e., of codimension 1 in $\cap_{k \in K} H_k$), and let $C_{m,K}$ be the remaining part. A star-subspace in $B_{m,K}$ is of the form $(\cap_{k \in K} H_k \cap H_j)^*$ for some index $j \in J \setminus K$. Then let R denote the set of such indices j , so that

$$|R| = |B_{m,K}|.$$

We consider two cases separately, depending on the dimension of the subspace $Y = \cap_{k \in K} H_k \cap D_l$.

Case 1. Y is a subspace of dimension $n - |K| - 1$. In this case, Y is of the form $(\cap_{k \in K} H_k) \cap H_y$ for some index y in I_{D_l} . It follows from (4.4), (4.5), (4.6) that the curve $f(\mathbb{C})$ lands inside the set

$$\cap_{k \in K} H_k \setminus \left(\cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K}) \right).$$

Now we need to show that this set is of the form (4.2). First, we verify the corresponding required inequality between cardinalities

$$\begin{aligned} |(J \setminus K) \setminus (R \cup \{y\})| &\geq |J \setminus K| - |B_{m,K}| - 1 \\ &\geq |J| - |J \cap K| - (|J| + 2|I| - 2n - 1 - |C_{m,K}|) - 1 \\ &= 2(n - |K|) + |C_{m,K}| + 2|K \setminus I| - |J \cap K| \\ &\geq 2(n - |K|) + 1 + |C_{m,K}|, \end{aligned}$$

where the last inequality holds because I and J are two disjoint sets and I is a proper subset of K . Secondly, we verify that the set K is of cardinality at most $n - 2$. Indeed, if $|K| = n - 1$, then since S is in general position with respect to $\{H_i\}_{1 \leq i \leq 2n}$, we see that

$$\cap_{k \in K} H_k \setminus \left(\cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K}) \right) = \cap_{k \in K} H_k \setminus \left(\cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \setminus C_{m,K} \right).$$

Owing to the inequality $|(J \setminus K) \setminus (R \cup \{y\})| \geq 3 + |C_{m,K}|$, the curve f lands in a complement of at least three points in a line. By Picard's theorem, f is constant, which is a contradiction.

Case 2. Y is a subspace of dimension at most $n - |K| - 2$. In this case, the curve $f(\mathbb{C})$ lands inside

$$\cap_{k \in K} H_k \setminus \left(\cup_{j \in (J \setminus K) \setminus R} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K} \cup Y^*) \right).$$

This set is of the form (4.2) since

$$|\{j \in (J \setminus K) \setminus R\}| \geq 2(n - |K|) + 1 + |C_{m,K} \cup Y^*|,$$

which also implies $|K| \leq n - 2$ by similar argument as in **Case 1**.

The lemma is thus proved. \square

End of proof of the Main Theorem. We now come back to the proof of the Main Theorem. Keep the notation as in Lemma 4.1. We claim that $\{\cap_{i \in I} H_i \cap H_j\}_{j \in J}$ is also a family of generic hyperplanes in the projective space $\cap_{i \in I} H_i \cong \mathbb{P}^{n-|I|}(\mathbb{C})$. Indeed, let $\mathcal{I}, \mathcal{J}, \mathcal{J}_1, \dots, \mathcal{J}_k$ be disjoint subsets of J such that $|\mathcal{I}|, |\mathcal{J}_i| \geq 2$, $|\mathcal{I}| + |\mathcal{J}_i| = (n - |I|) + 2$, $1 \leq i \leq k$ and let $\{i_1, \dots, i_l\}$ be a subset of \mathcal{I} . Let us set $\mathbf{I} = I \cup \mathcal{I}$; then the intersection between the $|\mathcal{J}|$ hyperplanes $H_j, j \in \mathcal{J}$, the k diagonal hyperplanes $H_{\mathbf{I}\mathcal{J}_1}, \dots, H_{\mathbf{I}\mathcal{J}_k}$, and the $|I| + l$ hyperplanes $H_i (i \in I), H_{i_1}, \dots, H_{i_l}$ is a linear subspace of codimension $\min\{k + |I| + l, |\mathbf{I}|\} + |\mathcal{J}|$, with the convention that when $\min\{k + |I| + l, |\mathbf{I}|\} + |\mathcal{J}| > n$, this intersection is empty. Since

$$\min\{k + |I| + l, |\mathbf{I}|\} + |\mathcal{J}| = \min\{k + l, |\mathcal{I}|\} + |I| + |\mathcal{J}|$$

we deduce that in the projective space $\cap_{i \in I} H_i$, the intersection between the $|\mathcal{J}|$ hyperplanes $H_j, j \in \mathcal{J}$, the k diagonal hyperplanes $H_{\mathcal{I}\mathcal{J}_1}, \dots, H_{\mathcal{I}\mathcal{J}_k}$, and the l hyperplanes H_{i_1}, \dots, H_{i_l} is a linear subspace of codimension $\min\{k + l, |\mathcal{I}|\} + |\mathcal{J}|$, with the convention that when $\min\{k + l, |\mathcal{I}|\} + |\mathcal{J}| > n - |I|$, this intersection is empty.

Starting point of the process by deformation: We start with the hyperbolicity of all complements of the forms

$$\cap_{i \in I} H_i \setminus \left(\cup_{j \in J} H_j \setminus A_{m, n-|I|} \right),$$

where $I, J, A_{m, n-|I|}$ are as in Lemma 4.1. More precisely,

- when $n = 3$, we start with the hyperbolicity of all complements $H_i \setminus \left(\cup_{j \neq i} H_j \right)$, which follows from Theorem 2.5 in $\mathbb{P}^2(\mathbb{C})$;
- when $n = 4$, we start with the hyperbolicity of all complements

$$\begin{aligned} & H_i \setminus \left(\cup_{j \neq i} H_j \right), \\ & \cap_{i \in I} H_i \setminus \left(\cup_{j \in J} H_j \setminus A_{1,2} \right) \quad (|I| = 2, 5 + |A_{1,2}| \leq |J| \leq 6), \end{aligned}$$

which follows from Theorem 2.5 in $\mathbb{P}^3(\mathbb{C})$ and Lemma 3.1 for $m = 1$;

- when $n = 5$, we start with the hyperbolicity of all complements

$$\begin{aligned} & H_i \setminus \left(\cup_{j \neq i} H_j \right), \\ & \cap_{i \in I} H_i \setminus \left(\cup_{j \in J} H_j \setminus A_{1,3} \right) \quad (|I| = 2, 7 + |A_{1,3}| \leq |J| \leq 8), \\ & \cap_{i \in I} H_i \setminus \left(\cup_{j \in J} H_j \setminus A_{2,2} \right) \quad (|I| = 3, 5 + |A_{2,2}| \leq |J| \leq 7), \end{aligned}$$

which follows from Theorem 2.5 in $\mathbb{P}^4(\mathbb{C})$, Lemma 3.2 for $m = 1$, and Lemma 3.1 for $m = 2$;

- when $n = 6$, we start with the hyperbolicity of all complements

$$\begin{aligned}
& H_i \setminus \left(\bigcup_{j \neq i} H_j \right), \\
& \bigcap_{i \in I} H_i \setminus \left(\bigcup_{j \in J} H_j \setminus A_{1,4} \right) \quad (|I| = 2, 9 + |A_{1,4}| \leq |J| \leq 10), \\
& \bigcap_{i \in I} H_i \setminus \left(\bigcup_{j \in J} H_j \setminus A_{2,3} \right) \quad (|I| = 3, 7 + |A_{2,3}| \leq |J| \leq 9), \\
& \bigcap_{i \in I} H_i \setminus \left(\bigcup_{j \in J} H_j \setminus A_{3,2} \right) \quad (|I| = 4, 5 + |A_{3,2}| \leq |J| \leq 8),
\end{aligned}$$

which follows from Theorem 2.5 in $\mathbb{P}^5(\mathbb{C})$, Lemma 3.3 for $m = 1$, Lemma 3.2 for $m = 2$, and Lemma 3.1 for $m = 3$.

Details of the process by deformation: In the first step, we apply inductively Lemma 4.1 for $l = n - 1$ and get at the end a hypersurface S_1 such that all complements of the forms

$$\begin{aligned}
& \bigcap_{i \in I} H_i \setminus \left(\bigcup_{j \in J} H_j \setminus (S_1 \cup A_{m,n-|I|}) \right) \quad (|I| = n-2), \\
& \bigcap_{i \in I} H_i \setminus \left(\bigcup_{j \in J} H_j \setminus ((\overline{\Delta}_{n-1} \cap S_1) \cup A_{m,n-|I|}) \right) \quad (|I| \leq n-3)
\end{aligned}$$

are hyperbolic. Considering this as the starting point of the second step, we apply inductively Lemma 4.1 for $l = n - 2$. Continuing this process, we get at the end of the $(n - 2)$ th step a hypersurface $S = S_{n-2}$ satisfying the required properties. \square

5 Some discussion

Actually, our method works for a family of at least $2n$ generic hyperplanes in $\mathbb{P}^n(\mathbb{C})$. We hope that the Main Theorem is true for all $n \geq 3$. As we saw above, the problem reduces to proving the following conjecture.

Conjecture. *All complements of the form (3.2) are hyperbolic.*

We already know it to be true for $n = 2$, since Lemma 3.1 holds generally, without restriction on m .

Lemma 5.1. *In $\mathbb{P}^2(\mathbb{C})$, all complements of the form (3.2) are hyperbolic*

Proof. Assume now $m \geq 4$ and $A_{m,2} = \{A_1, \dots, A_m\}$, where $A_i = H_{i_1} \cap H_{i_2}$ ($1 \leq i \leq m$). We denote by I the index set $\{i_j : 1 \leq i \leq m, 1 \leq j \leq 2\}$. Suppose to the contrary that there exists an entire curve $f: \mathbb{C} \rightarrow \mathbb{P}^2(\mathbb{C}) \setminus (\bigcup_{i=1}^{5+m} H_i \setminus A_{m,2})$. By the generic condition, we can assume that f is linearly nondegenerate. By similar arguments as in Lemma 3.1 (cf. (3.6)), we have

$$\sum_{i \in I} N_f^{[2]}(r, H_i) \leq 3 \sum_{i=1}^m N_f(r, A_i).$$

Let $\mathcal{C}_m = \{c_m = 0\}$ be an algebraic curve in $\mathbb{P}^2(\mathbb{C})$ of degree d passing through all points in $A_{m,2}$ with multiplicity at least k which does not contain the curve $f(\mathbb{C})$. Starting from the inequality

$$\min_{1 \leq j \leq 2} \text{ord}_z(h_{i_j} \circ f) \leq \frac{1}{k} \text{ord}_z(c_m \circ f) \quad (z \in f^{-1}(A_i))$$

and proceeding as in (3.8), we get

$$\sum_{i=1}^m N_f(r, A_i) \leq \frac{1}{k} N_f(r, \mathcal{C}_m).$$

We may then proceed similarly as in (3.9)

$$\begin{aligned}
(m+2)T_f(r) &\leq \sum_{i=1}^{5+m} N_f^{[2]}(r, H_i) + S_f(r) \\
&\leq 3 \sum_{i=1}^m N_f(r, A_i) + S_f(r) \\
&\leq \frac{3}{k} N_f(r, \mathcal{C}_m) + S_f(r) \\
&\leq \frac{3d}{k} T_f(r) + S_f(r).
\end{aligned} \tag{5.1}$$

When $m \geq 5$, the following claim yields a concluding contradiction.

Claim 5.1. *If $m \geq 5$, we can find some curve \mathcal{C}_m which does not contain $f(\mathbb{C})$ such that*

$$k > \frac{3d}{m+2}. \tag{5.2}$$

Indeed, the degree of freedom for the choice of a curve of degree d is

$$\frac{(d+1)(d+2)}{2} - 1.$$

We want \mathcal{C}_m to pass through all points in $A_{m,2}$ with multiplicity at least k . The number of equations (with the coefficients of \mathcal{C}_m as unknowns) for this is not greater than

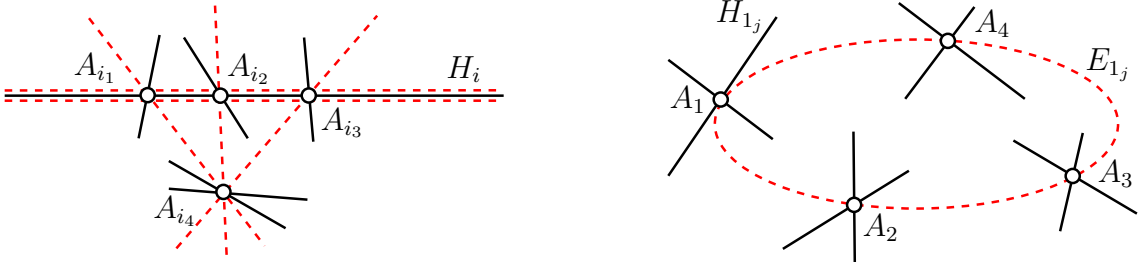
$$m \frac{k(k+1)}{2}.$$

Thus, for the existence of \mathcal{C}_m , it is necessary that

$$\frac{(d+1)(d+2)}{2} - 1 > m \frac{k(k+1)}{2}. \tag{5.3}$$

We try to find two natural numbers k, d satisfying (5.2) and (5.3). This can be done by choosing $d = (m+2)M$ and $k = 3M+1$ for large enough M . Using the remaining freedom in the choice of \mathcal{C}_m , we can choose it not containing $f(\mathbb{C})$, which proves the claim.

Next, we consider the remaining case where $m = 4$.



If there exists a collinear subset $\{A_{i_1}, A_{i_2}, A_{i_3}\}$ of $A_{4,2}$, then by the generic condition, it must be contained in some line H_i . Let A_{i_4} be the remaining point of the set $A_{4,2}$ and let \mathcal{C}_4 be the degenerate quintic consisting of the three lines $A_{i_j}A_{i_4}$ ($1 \leq j \leq 3$) and of the line H_i with multiplicity 2. Since \mathcal{C}_4 passes through all points in $A_{4,2}$ with multiplicity at least 3, the inequality (5.2) is satisfied. By using (5.1), we get a contradiction.

Now we assume that any subset of $A_{4,2}$ containing three points is not collinear. Let $E_{ij} = \{e_{ij} = 0\}$ ($1 \leq i \leq 4, 1 \leq j \leq 2$) be the eight conics passing through all points of $A_{4,2}$, tangent to the line H_{ij} at the point A_i ($1 \leq i \leq 4, 1 \leq j \leq 2$). Let $\mathcal{E} = \{e = 0\}$ be the degenerate curve of degree 16 consisting of all these E_{ij} . We claim that f does not land in \mathcal{E} . Otherwise, it lands in some conic E_{ij} . Since the number of intersection points between E_{ij} and $\cup_{i=1}^9 H_i \setminus A_{4,2}$ is > 3 and since any complement of three distinct points in an irreducible curve is hyperbolic, f must be constant, which is a contradiction.

Letting z be a point in $f^{-1}(A_i)$, we have

$$\text{ord}_z(\mathbf{e}_{i_j} \circ f) \geq 1 \quad (1 \leq i \leq 4, 1 \leq j \leq 2).$$

By the construction of E_{i_j} , if $\text{ord}_z(h_{i_j} \circ f) \geq 2$ for some $1 \leq j \leq 2$, then we also have $\text{ord}_z(\mathbf{e}_{i_j} \circ f) \geq 2$. Furthermore, if $\text{ord}_z(h_{i_j} \circ f) \geq 2$ for all $1 \leq j \leq 2$, then $\text{ord}_z(\mathbf{e}_{i_j} \circ f) \geq 2$ for all $1 \leq i \leq 4, 1 \leq j \leq 2$. Thus, the following inequality holds:

$$\begin{aligned} \min \{ \text{ord}_z(h_{i_1}) \circ f, 2 \} + \min \{ \text{ord}_z(h_{i_2}) \circ f, 2 \} &\leq \frac{1}{3} \sum_{i=1}^4 \sum_{j=1}^2 \text{ord}_z(\mathbf{e}_{i_j} \circ f) \\ &= \frac{1}{3} \text{ord}_z(\mathbf{e} \circ f) \quad (z \in f^{-1}(A_i)). \end{aligned}$$

This implies

$$\sum_{i \in I} N_f^{[2]}(r, H_i) \leq \frac{1}{3} N_f(r, \mathcal{E}).$$

We proceed similarly as before to derive a contradiction

$$\begin{aligned} 6 T_f(r) &\leq \sum_{i=1}^9 N_f^{[2]}(r, H_i) + S_f(r) \\ &\leq \frac{1}{3} N_f(r, \mathcal{E}) + S_f(r) \\ &\leq \frac{16}{3} T_f(r) + S_f(r). \end{aligned}$$

Lemma 5.1 is thus proved. □

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